Snowmass2021 - Letter of Interest

Beyond Pairs: Using Higher-Order Statistics to Illuminate the Universe's Contents and Laws

Thematic Areas:
■ (CF4) Dark Energy and Cosmic Acceleration: The Modern Universe
■ (CF5) Dark Energy and Cosmic Acceleration: Cosmic Dawn and Before
□ (CF6) Dark Energy and Cosmic Acceleration: Complementarity of Probes and New Facilities
■ (CF7) Cosmic Probes of Fundamental Physics
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Contact Information:
Zachary Slepian (University of Florida) [zslepian@ufl.edu]
Gustavo Niz (Universidad de Guanajuato) [g.niz@ugto.mx]

Authors: Alejandro Aviles (CONACyT/ININ), Florian Beutler (University of Edinburgh), Enzo Branchini (University of Rome), Phil Bull (University of London), Blakesley Burkhart (Flatiron/Rutgers), Robert Cahn (LBNL), Robert Caldwell (Dartmouth), Vincent Desjacques (Technion), Cora Dvorkin (Harvard), Alexander Eggemeier (Durham University), Daniel Eisenstein (Harvard), Simone Ferraro (LBNL), Andreu Font-Ribera (Institut de Fisica d'Altes Energies, The Barcelona Institute of Science and Technology), Jim Fry (University of Florida), Simon Foreman (Perimeter), Hector Gil-Marin (University of Barcelona), Jahmour Givans (The Ohio State University), ChangHoon Hahn (Princeton), Anthony Gonzalez (University of Florida), Oliver Hahn (Observatoire de la Cote d'Azur), Chris Hirata (The Ohio State University), Jiamin Hou (University of Florida), Donghui Jeong (The Pennsylvania State University), Alex Krolewski (University of Waterloo), Suman Majumdar (Indian Institute of Technology, Indore), Azadeh Moradinezhad Dizgah (University of Geneva), Mark Neyrinck (University of the Basque Country), Nikhil Padmanabhan (Yale), Oliver Philcox (Princeton), Stephen Portillo (University of Washington), Jonathan Pritchard (Imperial College), Ashley Ross (The Ohio State University), Cristiano Sabiu (Yonsei University), Shun Saito (Missouri University of Science and Technology), Lado Samushia (Kansas State University), Marcel Schmittfull (Institute for Advanced Study), Roman Scoccimarro (New York University), Sarah Shandera (The Pennsylvania State University), Makana Silva (Ohio State University), David Spergel (Flatiron), Istvan Szapudi (Institure for Astronomy, Hawaii), Atsushi Taruya (Kyoto University), Tejaswi Venumadhav (UC Santa Barbara), Francisco Villaescusa-Navarro (Princeton), Digvijay Wadekar (New York University), David Weinberg (The Ohio State University), Yvonne Y.Y. Wong (University of New South Wales), Idit Zehavi (Case Western Reserve), Matias Zaldarriaga (Institute for Advanced Study), Miguel Zumalacarregui (Leibniz-Institute for Astrophysics Potsdam)

Abstract: The coming decade will see factor of 100 expansions in data volume from both photometric (LSST/VRO) and spectroscopic (DESI, Euclid, NGRST/WFIRST, SPHEREx) surveys to probe fundamental physics, such as dark energy, gravity, inflation or neutrinos. This demands exploiting highly sophisticated analysis tools, both to extract maximal information and to provide cross-checks against systematics. To do so we suggest fully exploiting higher-order correlations in configuration space and polyspectra in Fourier space in the same way 2-point correlation function and power spectrum are currently used. We outline a number of science areas that can be impacted by higher-order statistics and also the challenges that must be addressed.

While galaxy surveys measure positions of individual objects, the underlying theories predict only statistical correlations between objects, not their absolute positions. Hence we must distill these datasets down to only relative information to enable comparison with theory models. For large-scale structure (LSS), measuring pair correlations in physical space (2-point correlation function, 2PCF) or Fourier space (power spectrum) is a well-mapped road to extracting information in surveys down to very small scales. Much work has been done on 2PCF and power spectrum, but the analogs for triplets, 3PCF and bispectrum, have been far less used due to computational expense; challenges regarding modeling the signal, its statistical errors, and systematics; and optimal weighting of two-point and three-point statistics. Use of correlations beyond triplets has been even sparser, with just two studies of the 4PCF[1;2] to date.

On large scales that have evolved linearly since the Universe's beginning, the matter density field is well described by the 2PCF/power spectrum. However, gravitational evolution of the matter density, which is bottom-up, induces higher-order correlations as one goes to smaller scales. Furthermore, on all scales, the distribution of $galaxies$ has higher-order correlations because galaxies do not perfectly trace the matter (encoded in biasing: they can tracer the matter density's square, and its tidal tensor, among others). Consequently, these correlations offer information both on the underlying physics—cosmological parameters, neutrino mass, etc.—and models of galaxy formation.

With the next generation of surveys, use of statistics beyond the 2PCF/power spectrum is likely to offer significant leverage on fundamental cosmological questions, and will do so for no additional cost in time in orbit or at the telescope. Realizing these gains requires only modest additional computational resources, but perhaps significant personpower for modeling and analysis. We now discuss science goals in rough order of physical scale.

Key Science Goals:

- **Reionization** The reionization of the Universe by the first stars around $z \sim 15$ is thought to involve rapidly expanding ionized bubbles, but their size distribution comes down to details of the gas density distribution and the feedback physics. These bubbles imprint rich non-Gaussian structure on the 21 cm signal from this epoch, and in turn 3PCF/bispectrum can be used to probe the bubble size and physics.[3–5] This can be applied to the large datasets expected from e.g., Square Kilometer Array (SKA) and HERA, to sharpen our picture of reionization physics for little additional cost compared to the significant outlay of constructing the arrays. Understanding reionization is also key for fundamental physics. Detecting the neutrino mass with the galaxy 2PCF/power spectrum requires the optical depth to reionization, currently measured from Planck CMB polarization with 10% precision. Improving our picture of reionization with 3PCF/bispectrum may offer a path to a better optical depth measurement (on a faster timescale than the Litebird CMB experiment) and hence a tighter neutrino mass constraint. Finally, in the pre-reionization ($z > 30$) era, the bispectrum is a treasure trove of information since the 21cm fluctuations are linear down to very small scales.[6]

- **Gravity** The UV-completion of GR remains an open problem in fundamental physics, and it is also possible that a low-energy modification to GR is in fact the source of the Universe's observed accelerated expansion. The 3PCF/bispectrum and beyond offer fertile ground for probing these extensions, since they stem partly from gravitational collapse. In particular, the kernel in perturbation theory producing them changes for different theories of modified gravity (MG),[7] as does f, the logarithmic derivative of the growth factor, which enters redshift-space distortions (RSD). Measuring the 3PCF/bispectrum can probe both, and adds significantly to the constraints by breaking the degeneracy between f and σ_8.[8–10] Relativistic effects in the bispectrum also enter one order sooner (in powers of scale/causal horizon scale) than they do in the power spectrum; detecting them well should be possible with upcoming surveys and will test GR on the largest possible scales.[11;12]

- **Expansion Rate Measurements** Both 3PCF and bispectrum have recently seen detections of BAO features usable as a standard ruler much as is already done with the 2PCF/power spectrum. In the absence of density field reconstruction, the 3PCF was equivalent to adding 1 years (out of 5) to SDSS BOSS; similar gains would be the case for e.g. DESI. While at the theory level, density field reconstruction pushes the 3PCF BAO information back into the 2PCF,[13] it remains important to use 3PCF as a complementary tool for BAO because it is methodologically independent and likely has different response to systematics. Furthermore, a large-scale, coherent, $z{\sim}1020$ baryon-dark matter relative velocity can imprint on galaxy formation so as to shift the BAO scale measured from the 2PCF. This bias can be constrained using the 3PCF, protecting the BAO method, and doing so will be vital for surveys such as DESI to achieve the desired accuracy.[14]

• **Neutrino Mass** Measuring the neutrino mass sum will fill in the last piece of the Standard Model, or rather, clarify the first piece of beyond-SM physics we have, including whether the neutrino mass hierarhy is normal or inverted. This measurement is very difficult from ground-based experiments; hence cosmology is the way forward. Because neutrinos free-stream at early times and slow down and behave like matter at low-z, they imprint a unique signature on LSS. This manifests as a decrement in the power spectrum beginning at quite large scales; thus to measure it one also needs CMB amplitude information, degenerate with the optical depth to reionization. Further, the power spectrum/2PCF are sensitive to galaxy biasing, which can mimic or mask neutrino mass. Adding the bispectrum/3PCF promises to greatly enhance our constraints on the neutrino mass.[15] Critically, the neutrino mass alters the $triangle\ angle$ dependence of the 3PCF, not just the scale dependence: this should aid in breaking the bias-mass degeneracy. Further, there are unique bispectrum neutrino signatures, such as an imaginary dipole due to neutrino-DM relative velocities, that can strengthen the mass constraint.[16]

• **Lyman-α forest** Spatial fluctuations of the UV background and temperature-density relation imprint rich structure in the forest that generates a 3PCF/bispectrum. A pathfinder study using simulations has recently been made,[17] and the enormous number of quasar sightlines available from DESI will offer an ideal dataset for the method. Since the UV background comes from rare, high-z quasars, probing it will expose how these massive, early objects form and evolve. This in turn supports their use for cosmology on large scales (relevant for primordial non-Gaussianity) and at early times (to pin down early dark energy models motivated by discrepancies in the high-z Hubble parameter[18] and by the H_0 tension[19]).

• **Primordial Non-Gaussianity (PNG)** The 3PCF/bispectrum are the lowest-order statistics sensitive to small deviations from a Gaussian Random Field (GRF) produced by inflation, testing physics at energy scales far beyond those probed in the laboratory.[20] The level of PNG detected will show if inflation was single-field or multi-field, and can also encode details about the particle content at that epoch (e.g. their spins).[21] While in detail PNG also does produce a power-spectrum signature,[22] there is potential degeneracy there to be broken by bispectrum.[23] PNG is a challenging measurement so using all tools available is critical, and bispectrum can improve constraints by $\sim 5\times$.[24] The volume offered by future 21 cm experiments (e.g. HIRAX) will also make them a key contributor to bispectrum-aided PNG constraints.[24–26]

• **Going Beyond Galaxy Triplets** All of the science goals discussed above can benefit from going beyond triplets of galaxies. Thus far there has been minimal use of 4PCF/trispectrum in LSS, but with new algorithms that render these computationally tractable, that should change for next-generation surveys. In addition, one-point cumulants of counts-in-cells, two-point cumulant correlators (skew spectra and beyond[27;28]), non-linear maps, such as sufficient statistics and its approximations, Box-Cox and logarithmic transformations,[29;30] and correlation of sliced fields[31;32] extract information beyond triplets with modest computational resources, even in redshift space. Other statistics aim to extract the information in triplets while still computing modified two-point statistics (e.g. weighted skew-spectra[33] and large-scale mode reconstruction[34]); improvements in 3PCF/bispectrum modelling will also improve predictions for these alternative statistics. Higher-point statistics in galaxy lensing and CMB lensing maps[35] also extract additional information from these observables. Last, machine learning methods[36] aim to capture all of the information in the density field beyond two-point statistics.

Challenges to Meet: Systematics must be well-controlled, especially to enable the large-scale measurements such as BAO, MG, neutrino mass, and PNG. Techniques exist for marginalization over selection biases[37] but using imaging data in concert (especially with LSST/VRO available) should further help. **Algorithmic** innovations are still needed: several fast solutions now exist for 3PCF/bispectrum,[38–41] including RSD,[42–44] but to go beyond triplets further development is needed (though one code exists for 4PCF[2]). These are computationally intensive algorithms, so increased performance from alternative computing architectures (e.g. GPUs) will be highly beneficial. Extracting maximal S/N from these measurements will require improved **modeling** of small, nonlinear scales; already higher-point measurements are limited by modelling rather than by statistics. Measuring e.g. 4PCF and 6PCF will directly probe the **covariance matrix** of 2PCF and 3PCF, as well, offering a more efficient path to optimally weighting the data than the current approach of using thousands of mocks.

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