Matching long and short distances in the form factors for Kaon decays

Giancarlo D'Ambrosio[1], Marc Knecht[2], and David Greynat[3]

[1]INFN Sezione di Napoli, Napoli (Italy)
[2]CNRS, CPT Marseille (France)
[3]Unlisted

August 2020

1 The general amplitude $\mathcal{A}^{K\to\pi\ell^+\ell^-}(s)$

Chiral perturbation theory describes QCD at low energies in particular kaon decays. Rare and radiative kaon decays have been extensively studied; for instance the CP-conserving decays $K(p) \to \pi(k)\ell^+\ell^-$ are dominated by the long-distance process $K \to \pi\gamma^*(q) \to \pi\ell^+\ell^-$ and the decay amplitudes can in general be written in terms of one form factor $W_i(z)$ $(i = \pm, S)$:

$$A\left(K^i \to \pi^i\ell^+\ell^-\right) = -\frac{e^2}{M_K^2(4\pi)^2}W_i(z)(k+p)^\mu \bar{u}_\ell(p_-)\gamma_\mu v_\ell(p_+) \ , \tag{1}$$

$z = q^2/M_K^2$; $W_i(z)$ can be decomposed as the sum of a polynomial piece plus a non-analytic term, $W_i^{\pi\pi}(z)$, generated by the $\pi\pi$ loop, completely determined in terms of the physical $K \to 3\pi$ amplitude. Keeping the polynomial terms up to $\mathcal{O}(p^6)$ we can write [1]

$$W_i(z) \,=\, G_F M_K^2\,(a_i \,+\, b_i z) \,+\, W_i^{\pi\pi}(z)\ , \tag{2}$$

where the parameters a_i and b_i parametrize local contributions starting respectively at $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ and fitted to the $K \to \pi\ell^+\ell^-$ data. For instance NA62 experiment collected physics data between 2016 and 2018 obtaining for $K^+ \to \pi^+\mu^+\mu^-$ [2]

$$a_+^{\mu\mu} = -0.592 \pm 0.015, \qquad b_+^{\mu\mu} = -0.699 \pm 0.058\ . \tag{3}$$

Our general program is to predict rare kaon decay amplitudes. In particular we have already studied the $K \to \pi\ell^+\ell^-$ amplitudes in eq. (1). So far, several results have been established [3,4]

1. we have obtained a general representation of $W_i^{\pi\pi}(z)$ accounting for all contributions due to pion exchanges at the two-loop level; we have shown that the representation proposed earlier in ref. [1] provides a very good description of the full result in the experimentally relevant range of z;

2. we have obtained an exact matching to short distance operators, which are i) four quark operators

$$\mathcal{L}^{\Delta S=1}_{\text{non-lept}}(x) = \left(-\frac{G_{\rm F}}{\sqrt{2}} V_{us} V_{ud}\right) \times \sum_{I=1}^{6} C_I(\nu) Q_I(x;\nu). \tag{4}$$

and ii) direct $\Delta S = 1$ leptonic operators

$$\mathcal{L}^{\Delta S=1}_{\text{lept}}(x;\nu) = -\frac{G_{\rm F}}{\sqrt{2}} V^*_{us} V_{ud} C_{7V}(\nu) Q_{7V}(x), \tag{5}$$

with $Q_{7V} = (\bar{s}^i d_i)_{V-A}(\bar{\ell}\ell)_V$. We then write the $s-$dependent amplitude as [3,4]

$$\begin{aligned}
\mathcal{A}^{K\to\pi\ell^+\ell^-}(s) &= -e^2 \times \bar{\rm u}(p_-)\gamma_\rho {\rm v}(p_+) \times \frac{1}{s} \times \left\{ i\int d^4x\, \langle \pi(p)| T\{j^\rho(0)\mathcal{L}^{\Delta S=1}_{\text{non-lept}}(x)\}|K(k)\rangle \right. \\
&\quad \left. - \left(\frac{G_{\rm F}}{\sqrt{2}} V_{us} V_{ud}\right) \times \frac{C_{7V}(\nu)}{4\pi\alpha} \times s \times \langle \pi(p)|(\bar{s}\gamma^\rho d)(0)|K(k)\rangle \right\} \\
&= -e^2 \times \bar{\rm u}(p_-)(\not{k} - \not{p}){\rm v}(p_+) \times \frac{W^{\rm LD}_{K\pi}(z;\nu) + W^{\rm SD}_{K\pi}(z;\nu)}{16\pi^2 M_K^2}
\end{aligned} \tag{6}$$

such that the dependence on the short-distance scale cancels between the two contributions, at leading [3] and at next-to-leading [4] orders in perturbative QCD;

3. we have proposed a research program based on a dispersive approach in order to obtain the long-distance part of $W^{\rm LD}_{K\pi}(z;\nu)$ and thus predict the parameters a_i and b_i through dispersion-relation sum rules. This aspect will be further developed in the next section;

4. we have also considered various fits to the data available at the time of ref. [3]. In particular, we have pointed out the possibility, in the case of $K^\pm \to \pi^\pm \ell\ell$ and given higher statistics, to perform a fit not only with respect of the parameters a_+ and b_+, but also to the slope β_+ of the $K \to \pi\pi\pi$ Dalitz-plot that enters in the expression of $W^{\pi\pi}_+(z)$ and which is only poorly determined (at the $\sim 35\%$ level) from global fits to the $K \to \pi\pi\pi$ Dalitz-plots;

2 The two-pion intermediate state contribution to $\mathcal{A}^{K\to\pi\ell^+\ell^-}(s)$

In eq. 6 we have written a decomposition of the amplitude in terms of two parts, dominated by long and by short distances, respectively. The determination of the latter is not problematic, since the relevant hadronic matrix element is available from the measurement of the form factors in $K_{\ell 3}$ decays. For the long-distance component, we have proposed in [3] a phenomemological approach based on a decomposition of the type

$$W^{\rm LD}_{K\pi}(z;\nu) = W^{\pi\pi}_{K\pi}(z) + W^{K\pi}_{K\pi}(z) + W^{K\bar{K}}_{K\pi}(z;\nu) + W^{\rm res}_{K\pi}(z;\nu), \tag{7}$$

where the intermediate states producing the lowest-lying thresholds, here $\pi\pi$, $K\pi$ and $K\bar{K}$, have been singled out, whereas the intermediate states corresponding to thresholds above ~ 1 GeV are described in an inclusive manner by an infinite set of narrow resonance states, in the spirit of a large-N_c representation of the matrix element. The residues and masses of these resonances have to be tuned in order to correctly reproduce the short-distance behaviour required by QCD, such as to correctly match the QCD scale dependence of the short-distance component. This aspect is well understood and has been given an appropriate treatment in refs. [3, 4]. The few explicit intermediate states with thresholds up to ~ 1 GeV, are to be treated in a dispersive manner, assuming unsubtracted dispersion relations.

A first illustration of this procedure has been given in [3], including only the elastic $\pi\pi$ P-wave channel (i.e. the ρ meson),

$$W_i^{\pi\pi}(z) = \int_{4M_\pi^2}^{\infty} dx \frac{\rho_i^{\pi\pi}(x)}{x - zM_K^2 - i0}. \tag{8}$$

The absorptive part consists of the two-pion spectral density $\rho_+^{\pi\pi}(s)$, and is obtained upon inserting a two-pion intermediate state in the representation of the form factor given in eq. 6, for instance

$$\begin{aligned} \rho_+^{\pi\pi}(s) \quad &= 16\pi^2 M_K^2 \times \tfrac{s-4M_\pi^2}{s}\,\theta(s-4M_\pi^2) \times F_V^\pi(s) \times \\ &\qquad \frac{f_1^{K^\pm\pi^\mp\to\pi^+\pi^-}(s)}{\lambda_{K\pi}^{1/2}(s)}. \end{aligned} \tag{9}$$

The pion form factor $F_V^\pi(s)$ and the P-wave projection of the scattering amplitude for $K^\pm\pi^\mp \to \pi^+\pi^-$ were constructed by a simple unitarization procedure from their known low-energy expansions. Efforts towards improving on this simple approach are underway. Once the dispersive machinery is available, other intermediate states, like $K\pi$ or even $K\bar{K}$ will also be considered, in a multi-channel approach.

3 Rare kaon decay modes induced by two-photon exchanges

The rare kaon decays studied in the preceding sections proceed through the exchange of a single virtual photon, $K \to \pi\gamma^* \to K\pi\ell^+\ell^-$. We conclude this short notes by briefly mentioning that we are also planning to investigate the possibility of addressing rare kaon decays proceeding through the exchange of two virtual photons with a similar approach. Considering for instance the case of $K_{S,L} \to \ell^+\ell^-$, we have again a decomposition of the amplitude into a short-distance dominated part [5] [a similar expression can be written with K^0 replaced by $\bar{K}^9$],

$$\mathcal{A}_{\rm SD}(K^0 \to \ell^+\ell^-) = -\sqrt{2}\,G_F m_\ell \frac{\alpha(M_Z)}{\pi \sin^2\theta_{\rm w}}\,\bar{\rm u}(p_-)i\gamma_5{\rm v}(p_+)F_K\,[V_{ts}^* V_{td} Y(x_t) + V_{cs}^* V_{cd} Y_{\rm NL}]\,, \tag{10}$$

and a part dominated by long distances, which can be written as

$$
\begin{aligned}
&\mathcal{A}_{\rm LD}(K^0 \to \ell^+\ell^-) \ = \\
&= \ -e^2\left(-\frac{G_F}{\sqrt{2}}V_{ud}V_{us}^*\right)\int\frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2}\,\frac{1}{(k-q)^2}\,\bar{\rm u}(p_-)\gamma^\rho\frac{1}{[\not p_- - \not q - m_\ell]}\gamma^\tau{\rm v}(p_+) \\
&\quad\times\int d^4x\int d^4y\, e^{iq\cdot y}\,e^{i(k-q)\cdot x}\Big[\sum_I C_I(\nu)ie^2\,\langle 0|T\{j_\rho(y)j_\tau(x)Q_I(0;\nu)\}|K^0(k)\rangle \\
&\qquad\qquad - (k-q)^2\delta^4(x)C_{7V}(\nu)\,\langle 0|T\{j_\rho(y)[\bar{s}\gamma_\tau(1-\gamma_5)d](0)\}|K^0(k)\rangle \\
&\qquad\qquad - q^2\delta^4(y)C_{7V}(\nu)\,\langle 0|T\{j_\tau(x)[\bar{s}\gamma_\rho(1-\gamma_5)d](0)\}|K^0(k)\rangle\Big] \\
&\quad + e^2\left(-\frac{G_F}{\sqrt{2}}V_{ud}V_{us}^*\right)C_{7A}\int\frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2}\,\bar{\rm u}(p_-)\gamma^\rho\frac{1}{[\not p_- - \not q - m_\ell]}\gamma^\tau\gamma_5{\rm v}(p_+) \\
&\quad\times\int d^4y\, e^{iq\cdot y}\,\langle 0|T\{j_\rho(y)[\bar{s}\gamma_\tau(1-\gamma_5)d](0)\}|K^0(k)\rangle \\
&\quad + e^2\left(-\frac{G_F}{\sqrt{2}}V_{ud}V_{us}^*\right)C_{7A}\int\frac{d^4q}{(2\pi)^4}\,\frac{1}{(k-q)^2}\,\bar{\rm u}(p_-)\gamma^\tau\gamma_5\frac{1}{[\not p_- - \not q - m_\ell]}\gamma^\rho{\rm v}(p_+) \\
&\quad\times\int d^4x\, e^{i(k-q)\cdot x}\,\langle 0|T\{j_\rho(x)[\bar{s}\gamma_\tau(1-\gamma_5)d](0)\}|K^0(k)\rangle. \qquad (11)
\end{aligned}
$$

This expression has been written such as to make the role played by the operator Q_{7V} and its Wilson coefficient clear. The time-ordered product of the electromagnetic current with the four-quark operators is singular at short-distances, and needs to be renormalized. The corresponding singularity is absorbed by the bare coefficient C_{7V}, which becomes renormalized. After this renormalization, both the time-ordered product and C_{7V} depend on the renormalization scale ν. The task that needs to be done is to simultaneously obtain a determination of the hadronic matrix elements appearing in this expression, and to correctly match the different contributions such that the sum does no longer depend, at a given order, on the renormalization scale ν.

References

[1] G. D'Ambrosio, G. Ecker, G. Isidori and J. Portoles, JHEP **08** (1998), 004 doi:10.1088/1126-6708/1998/08/004 [arXiv:hep-ph/9808289 [hep-ph]].

[2] NA62 Collaboration *New measurement of the $K^+ \to \pi^+\mu^+\mu^-$ decay at NA62* presented by Lubos Bician at 40th International Conference on High Energy Physics (ICHEP2020) Prague, Czech Republic, 30 Jul - 05 Aug 2020

[3] G. D'Ambrosio, D. Greynat and M. Knecht, JHEP **02** (2019), 049 doi:10.1007/JHEP02(2019)049 [arXiv:1812.00735 [hep-ph]].

[4] G. D'Ambrosio, D. Greynat and M. Knecht, Phys. Lett. B **797** (2019), 134891 doi:10.1016/j.physletb.2019.134891 [arXiv:1906.03046 [hep-ph]].

[5] G. Buchalla and A. J. Buras, *The rare decays $K^+ \to \pi^+\nu\bar{\nu}$ and $K_L \to \mu^+\mu^-$ beyond leading logarithms*, Nucl. Phys. B **412**, 106 (1994) [hep-ph/9308272].