String Perturbation Theory: Letter of Interest for Snowmass 2021

Nathan Berkovits,[1,*] Eric D'Hoker,[2,†] Michael B. Green,[3,4,‡] Henrik Johansson,[5,6,§] and Oliver Schlotterer[5,¶]

[1] *ICTP South American Institute for Fundamental Research, Instituto de Física Teórica, UNESP-Universidade Estadual Paulista, Sao Paulo 01140-070, SP, Brazil*

[2] *Mani L. Bhaumik Institute for Theoretical Physics, Department of Physics and Astronomy, University of California, Los Angeles, CA 90095, USA*

[3] *Department of Applied Mathematics and Theoretical Physics, Cambridge CB3 0WA, UK*

[4] *School of Physics and Astronomy, Queen Mary University of London, London, E1 4NS, UK*

[5] *Department of Physics and Astronomy, Uppsala University, 75108 Uppsala, Sweden*

[6] *Nordita, Stockholm University and KTH Royal Institute of Technology, SE-10691 Stockholm, Sweden*

(Dated: March 10, 2021)

String theory unifies gravity and Yang-Mills theory in a UV finite quantum theory that is expected to shed light on the dynamics of black holes and the early universe. The theory may be approached from a number of complementary angles. String perturbation theory is a topological expansion in powers of the string coupling g_s obtained by summing over random fluctuating surfaces. The low energy expansion in powers of the string scale α' is given by supergravity plus higher order string corrections to supergravity encoded in local effective interactions. Supergravity is valid for low energy but holds for all values of g_s while string perturbation theory is valid for small g_s but holds for all energies. The gauge/gravity correspondence formulates string theory in certain hyperbolic space-times via correlators in Yang-Mills theory. At the intersection of these approaches, effective interactions are calculable in string perturbation theory or in the gauge/gravity correspondence, and may be able to predict physical effects observable at low energy.

Superstring perturbation theory has grown into a rich subject whose study has revealed deep connections with quantum field theory amplitudes, D-brane and black hole dynamics, gauge/gravity duality, algebraic geometry and modular forms. At the heart of superstring perturbation theory are superstring amplitudes – primarily considered for massless gravitons, gauge bosons and their respective supersymmetry partners (see for example [1, 2] for videos of recent lectures). Following the discovery of Type I [3], Type II [4], and Heterotic strings [5], much of the formalism of superstring perturbation theory dates back to the 1980's (see [6, 7] and references therein).

The developments of the past 20 years have dramatically advanced the explicit computation of a significant number of string amplitudes, and sharpened our understanding of the formalism. This progress was driven by a confluence of ideas from different formulations of the superstring, including the Ramond-Neveu-Schwarz (RNS) formalism [8, 9] and the pure-spinor formalism [10–12], whose interrelation was clarified in [13]. In particular, the two-loop four-graviton amplitude was computed in the RNS formulation in [14] and extended to include external fermions in the pure-spinor formulation [15, 16]. The mathematical structure of the RNS formulation, including the role of supermoduli, was revisited and substantially clarified in [17–19]. Two-loop contributions to the cosmological constant in orbifold compactifications were evaluated in [20, 21].

Many of these results enjoy unexpected simplicity and exhibit a wealth of hidden structure related to field-theory amplitudes. In particular, the interplay between open and closed string amplitudes, illustrated by the Kawai-Lewellen-Tye (KLT) relations at tree-level [22] and chiral splitting in terms of internal loop momenta at arbitrary loop order [23], are intimately connected with the double-copy construction of supergravity amplitudes in terms of their Yang-Mills counterparts [24]. This connection was shown in [25, 26] to underly the double-copy construction of many supergravity amplitudes [27, 28] and stimulated the construction of the two-loop five-point function in [29]. Studies of the ultra-violet divergences in supergravity at five-loop order [30] were preceded by a corresponding analysis based on string theory [31].

Just as string theory elegantly explains far-reaching properties of field-theory amplitudes, the evaluation of string amplitudes also benefits from field-theory structures. For instance, the open superstring tree-level amplitudes themselves admit a double-copy construction [32] akin to the KLT relations for gravitational amplitudes [33] with first loop-level extensions proposed in [34]. The analogous double-copy constructions of the open bosonic string or gauge-gravity couplings of heterotic strings involve a massive gauge theory [35] whose study was motivated by conformal-supergravity amplitudes [36].

String perturbation theory provides powerful tests of some of the many dualities that relate different perturbative superstring theories [37, 38] (see [39] for an early overview). For example, $SL(2,\mathbb{Z})$ self-duality of Type IIB string theory requires the coefficients of low energy effective interactions to be modular functions or forms in the axion-dilaton field [40, 41], while space-time supersymmetry requires them to obey differential equations [42]. Torus compactifications of Type IIB superstrings give rise to higher-rank duality groups whose automorphic forms are reviewed in [43]. The resulting relations between the coefficients of various effective interaction were tested successfully against the perturbative superstring predictions to one, two, and three loop orders in [44, 45], [46–48] and [49], respectively.

The interplay of dualities and the low energy expansion of superstring amplitudes results in exciting and unexpected algebraic and arithmetic structure. Already at genus zero, the pattern of Multiple Zeta Values (MZVs) in higher-point functions is elegantly understood in terms of Hopf-algebra structures of motivic MZVs and the Drinfeld associator [50, 51]. Furthermore, the closed-string low-energy expansions at tree level can be obtained from open strings through a formal operation on MZVs [52–55], the so-called single-valued map. At one loop and beyond, string amplitudes produce a wealth of elliptic and modular generalizations of MZVs. Closely related elliptic polylogarithms not only drive open-string [56] and closed-string [57] computations but are also improving the techniques available to evaluate elliptic Feynman integrals [58, 59]. Uniform transcendentality, familiar from dimensionally regularized Feynman integrals, enjoys a string amplitude counterpart in its α' expansion at tree level [60, 61] and in [62–64] at one-loop order.

The low energy expansion of closed-string amplitudes at one-loop order and beyond has introduced fascinating families of non-holomorphic modular forms, dubbed modular graph functions [65–67] and forms [68, 69] (see [70] for an overview) which led to further research, for example in string theory [71–75] and in mathematics [76–78]. Their description via iterated integrals of holomorphic objects sheds new light on the relation between open and closed strings through the single-valued map [79, 80] and related single-valued integration [81, 82] and Betti-deRham duality [83, 84]. The study of higher-genus modular graph functions and forms [85] and their generalizations to tensors [86] has just begun and is expected to involve strong ties with open-string amplitudes, non-holomorphic Siegel modular forms [87], and new function spaces for Feynman integrals in quantum field theory.

Many questions related to string perturbation theory remain open to future exploration. An immediate goal is to push for explicit calculation of open and closed string amplitudes, both at higher multiplicity and higher loop order. Although a precise RNS formulation of superstring amplitudes is available in terms of integrals over supermoduli space, the intricate structure of this space [88] promises any practical calculations beyond two-loop order to be daunting. A further goal is to extract practical rules for supermoduli, possibly using vertical integration [89]. Although the manifest supersymmetry of the pure spinor formulation simplifies the evaluation of numerous superstring amplitudes, higher multiplicities and loop orders are marred by divergences in the λ-ghost integrals and the difficulty in evaluating non-trivial b-ghost correlators. Another goal is to resolve these divergences and find a method for evaluating the b-ghost correlators.

Concerning the interplay of open and closed strings, it will be valuable to search for loop-level generalizations of the KLT relations [90] and the single-valued map and to explore their implications for the double-copy structure of supergravity amplitudes. Relatedly, recent calculations of Yang-Mills correlators for finite gauge coupling in the large N limit [91, 92] promise to shed new light on the $SL(2,\mathbb{Z})$ duality of Type IIB strings via holography, and on the string uncertainty principle. Other goals include exploring the relation between string field theory and string perturbation theory, and gauge/gravity duality via string and Mellin amplitudes in AdS [93–96]. Finally, applications to black hole and gravitational wave physics will certainly gain further importance [97, 98].

* nathan.berkovits@unesp.br
† dhoker@physics.ucla.edu
‡ m.b.green@damtp.cam.ac.uk
§ henrik.johansson@physics.uu.se
¶ oliver.schlotterer@physics.uu.se

[1] E. D'Hoker, "Lecture Series on String Amplitudes," *Amplitudes 2018 Summer School*, UC Davis (2018).
[2] O. Schlotterer, "Lecture series on Superstring Amplitudes in the RNS and pure spinor formalism," *String theory from the worldsheet perspective*, GGI, Florence, Italy (2019).
[3] M.B. Green and J.H. Schwarz, "Supersymmetrical Dual String Theory," Nucl. Phys. B **181**, 502–530 (1981).
[4] M.B. Green and J.H. Schwarz, "Supersymmetrical String Theories," Phys. Lett. B **109**, 444–448 (1982).
[5] D.J. Gross, J.A. Harvey, E.J. Martinec, and R. Rohm, "The Heterotic String," Phys. Rev. Lett. **54**, 502–505 (1985).
[6] D. Friedan, E.J. Martinec, and S.H. Shenker, "Conformal Invariance, Supersymmetry and String Theory," Nucl. Phys. B **271**, 93–165 (1986).
[7] E. D'Hoker and D.H. Phong, "The Geometry of String Perturbation Theory," Rev. Mod. Phys. **60**, 917 (1988).
[8] E. D'Hoker and D.H. Phong, "Two loop superstrings. 1. Main formulas," Phys. Lett. B **529**, 241–255 (2002), arXiv:hep-th/0110247.
[9] E. D'Hoker and D.H. Phong, "Lectures on two loop superstrings," Conf. Proc. C **0208124**, 85–123 (2002), arXiv:hep-th/0211111.
[10] N. Berkovits, "Super Poincare covariant quantization of the superstring," JHEP **04**, 018 (2000), arXiv:hep-th/0001035.
[11] N. Berkovits, "Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring," JHEP **09**, 047 (2004), arXiv:hep-th/0406055.
[12] N. Berkovits, "Pure spinor formalism as an N=2 topological string," JHEP **10**, 089 (2005), arXiv:hep-th/0509120.
[13] N. Berkovits, "Untwisting the pure spinor formalism to the RNS and twistor string in a flat and $AdS_5 \times S^5$ background," JHEP **06**, 127 (2016), arXiv:1604.04617.
[14] E. D'Hoker and D.H. Phong, "Two-loop superstrings VI: Non-renormalization theorems and the 4-point function,"

Nucl. Phys. B **715**, 3–90 (2005), arXiv:hep-th/0501197.

[15] N. Berkovits, "Super-Poincare covariant two-loop superstring amplitudes," JHEP **01**, 005 (2006), arXiv:hep-th/0503197.

[16] N. Berkovits and C.R. Mafra, "Equivalence of two-loop superstring amplitudes in the pure spinor and RNS formalisms," Phys. Rev. Lett. **96**, 011602 (2006), arXiv:hep-th/0509234.

[17] E. Witten, "Notes On Super Riemann Surfaces And Their Moduli," Pure Appl. Math. Quart. **15**, 57–211 (2019), arXiv:1209.2459.

[18] E. Witten, "Superstring Perturbation Theory Revisited," (2012), arXiv:1209.5461.

[19] E. Witten, "Notes On Holomorphic String And Superstring Theory Measures Of Low Genus," (2013), arXiv:1306.3621.

[20] E. D'Hoker and D.H. Phong, "Two-loop vacuum energy for Calabi-Yau orbifold models," Nucl. Phys. B **877**, 343–400 (2013), arXiv:1307.1749.

[21] N. Berkovits and E. Witten, "Supersymmetry Breaking Effects using the Pure Spinor Formalism of the Superstring," JHEP **06**, 127 (2014), arXiv:1404.5346.

[22] H. Kawai, D.C. Lewellen, and S.H.H. Tye, "A Relation Between Tree Amplitudes of Closed and Open Strings," Nucl. Phys. **B269**, 1–23 (1986).

[23] E. D'Hoker and D.H. Phong, "Conformal Scalar Fields and Chiral Splitting on Superriemann Surfaces," Commun. Math. Phys. **125**, 469 (1989).

[24] Z. Bern, J.J.M. Carrasco, and H. Johansson, "Perturbative Quantum Gravity as a Double Copy of Gauge Theory," Phys. Rev. Lett. **105**, 061602 (2010), arXiv:1004.0476.

[25] C.R. Mafra, O. Schlotterer, and S. Stieberger, "Explicit BCJ Numerators from Pure Spinors," JHEP **07**, 092 (2011), arXiv:1104.5224.

[26] C.R. Mafra and O. Schlotterer, "Towards one-loop SYM amplitudes from the pure spinor BRST cohomology," Fortsch. Phys. **63**, 105–131 (2015), arXiv:1410.0668.

[27] Z. Bern, J.J.M. Carrasco, and H. Johansson, "New Relations for Gauge-Theory Amplitudes," Phys. Rev. **D78**, 085011 (2008), arXiv:0805.3993.

[28] Z. Bern, J.J.M. Carrasco, M. Chiodaroli, H. Johansson, and R. Roiban, "The Duality Between Color and Kinematics and its Applications," (2019), arXiv:1909.01358.

[29] E. D'Hoker, C.R. Mafra, B. Pioline, and O. Schlotterer, "Two-loop superstring five-point amplitudes. Part I. Construction via chiral splitting and pure spinors," JHEP **08**, 135 (2020), arXiv:2006.05270.

[30] Z. Bern, J.J.M. Carrasco, W.M. Chen, A. Edison, H. Johansson, J. Parra-Martinez, R. Roiban, and M. Zeng, "Ultraviolet Properties of $\mathcal{N} = 8$ Supergravity at Five Loops," Phys. Rev. D **98**, 086021 (2018), arXiv:1804.09311.

[31] J. Bjornsson and M.B. Green, "5 loops in 24/5 dimensions," JHEP **08**, 132 (2010), arXiv:1004.2692.

[32] C.R. Mafra, O. Schlotterer, and S. Stieberger, "Complete N-Point Superstring Disk Amplitude I. Pure Spinor Computation," Nucl. Phys. **B873**, 419–460 (2013), arXiv:1106.2645.

[33] J. Broedel, O. Schlotterer, and S. Stieberger, "Polylogarithms, Multiple Zeta Values and Superstring Amplitudes," Fortsch. Phys. **61**, 812–870 (2013), arXiv:1304.7267.

[34] C.R. Mafra and O. Schlotterer, "Double-Copy Structure of One-Loop Open-String Amplitudes," Phys. Rev. Lett. **121**, 011601 (2018), arXiv:1711.09104.

[35] T. Azevedo, M. Chiodaroli, H. Johansson, and O. Schlotterer, "Heterotic and bosonic string amplitudes via field theory," JHEP **10**, 012 (2018), arXiv:1803.05452.

[36] H. Johansson and J. Nohle, "Conformal Gravity from Gauge Theory," (2017), arXiv:1707.02965.

[37] C.M. Hull and P.K. Townsend, "Unity of superstring dualities," Nucl. Phys. **B438**, 109–137 (1995), [,236(1994)], arXiv:hep-th/9410167.

[38] E. Witten, "String theory dynamics in various dimensions," Nucl. Phys. **B443**, 85–126 (1995), arXiv:hep-th/9503124.

[39] N.A. Obers and B. Pioline, "U duality and M theory," Phys. Rept. **318**, 113–225 (1999), arXiv:hep-th/9809039.

[40] M.B. Green and M. Gutperle, "Effects of D instantons," Nucl. Phys. **B498**, 195–227 (1997), arXiv:hep-th/9701093.

[41] M.B. Green, M. Gutperle, and P. Vanhove, "One loop in eleven-dimensions," Phys. Lett. **B409**, 177–184 (1997), arXiv:hep-th/9706175.

[42] M.B. Green and S. Sethi, "Supersymmetry constraints on type IIB supergravity," Phys. Rev. D **59**, 046006 (1999), arXiv:hep-th/9808061.

[43] P. Fleig, H.P.A. Gustafsson, A. Kleinschmidt, and D. Persson, *Eisenstein series and automorphic representations*, Cambridge Studies in Advanced Mathematics, Vol. 176 (Cambridge University Press, Cambridge, 2018) arXiv:1511.04265.

[44] M.B. Green and P. Vanhove, "The Low-energy expansion of the one loop type II superstring amplitude," Phys.Rev. **D61**, 104011 (2000), arXiv:hep-th/9910056.

[45] M.B. Green, J.G. Russo, and P. Vanhove, "Low energy expansion of the four-particle genus-one amplitude in type II superstring theory," JHEP **02**, 020 (2008), arXiv:0801.0322.

[46] E. D'Hoker, M. Gutperle, and D.H. Phong, "Two-loop superstrings and S-duality," Nucl. Phys. B **722**, 81–118 (2005), arXiv:0503180.

[47] E. D'Hoker, M.B. Green, B. Pioline, and R. Russo, "Matching the D^6R^4 interaction at two-loops," JHEP **01**, 031 (2015), arXiv:1405.6226.

[48] E. D'Hoker, C.R. Mafra, B. Pioline, and O. Schlotterer, "Two-loop superstring five-point amplitudes. Part II. Low energy expansion and S-duality," JHEP **02**, 139 (2021), arXiv:2008.08687.

[49] H. Gomez and C.R. Mafra, "The closed-string 3-loop amplitude and S-duality," JHEP **10**, 217 (2013), arXiv:1308.6567.

[50] O. Schlotterer and S. Stieberger, "Motivic Multiple Zeta Values and Superstring Amplitudes," J. Phys. **A46**, 475401 (2013), arXiv:1205.1516.

[51] J.M. Drummond and E. Ragoucy, "Superstring amplitudes and the associator," JHEP **08**, 135 (2013), arXiv:1301.0794.

[52] S. Stieberger and T.R. Taylor, "Closed String Amplitudes as Single-Valued Open String Amplitudes," Nucl. Phys. **B881**, 269–287 (2014), arXiv:1401.1218.

[53] O. Schlotterer and O. Schnetz, "Closed strings as single-valued open strings: A genus-zero derivation," J. Phys. **A52**, 045401 (2019), arXiv:1808.00713.

[54] P. Vanhove and F. Zerbini, "Closed string amplitudes from single-valued correlation functions," (2018), arXiv:1812.03018.

[55] F. Brown and C. Dupont, "Single-valued integration and superstring amplitudes in genus zero," (2019), arXiv:1910.01107.

[56] J. Broedel, C.R. Mafra, N. Matthes, and O. Schlotterer, "Elliptic multiple zeta values and one-loop superstring amplitudes," JHEP **07**, 112 (2015), arXiv:1412.5535.

[57] E. Panzer, "Talk on modular graph functions as iterated Eisenstein integrals," *Elliptic Integrals in Mathematics and Physics*, Ascona, Switzerland (2018).

[58] L. Adams and S. Weinzierl, "Feynman integrals and iterated integrals of modular forms," Commun. Num. Theor. Phys. **12**, 193–251 (2018), arXiv:1704.08895.

[59] J. Broedel, C. Duhr, F. Dulat, and L. Tancredi, "Elliptic polylogarithms and iterated integrals on elliptic curves. Part I: general formalism," JHEP **05**, 093 (2018), arXiv:1712.07089.

[60] J.M. Henn, "Multiloop integrals in dimensional regularization made simple," Phys. Rev. Lett. **110**, 251601 (2013), arXiv:1304.1806.

[61] J. Broedel, O. Schlotterer, S. Stieberger, and T. Terasoma, "All order α'-expansion of superstring trees from the Drinfeld associator," Phys. Rev. **D89**, 066014 (2014), arXiv:1304.7304.

[62] L. Adams and S. Weinzierl, "The ε-form of the differential equations for Feynman integrals in the elliptic case," Phys. Lett. **B781**, 270–278 (2018), arXiv:1802.05020.

[63] E. D'Hoker and M.B. Green, "Exploring transcendentality in superstring amplitudes," JHEP **07**, 149 (2019), arXiv:1906.01652.

[64] C.R. Mafra and O. Schlotterer, "All-order alpha'-expansion of one-loop open-string integrals," Phys. Rev. Lett. **124**, 101603 (2020), arXiv:1908.09848.

[65] E. D'Hoker, M.B. Green, and P. Vanhove, "On the modular structure of the genus-one Type II superstring low energy expansion," JHEP **08**, 041 (2015), arXiv:1502.06698.

[66] F. Zerbini, "Single-valued multiple zeta values in genus 1 superstring amplitudes," Commun. Num. Theor. Phys. **10**, 703–737 (2016), arXiv:1512.05689.

[67] E. D'Hoker, M.B. Green, Ö. Gürdogan, and P. Vanhove, "Modular graph functions," Commun. Num. Theor. Phys. **11**, 165–218 (2017), arXiv:1512.06779.

[68] E. D'Hoker and M.B. Green, "Identities between modular graph forms," J. Number Theory **189**, 25–80 (2018), arXiv:1603.00839.

[69] E. D'Hoker and J. Kaidi, "Hierarchy of Modular Graph Identities," JHEP **11**, 051 (2016), arXiv:1608.04393.

[70] J.E. Gerken, "Modular Graph Forms and Scattering Amplitudes in String Theory," (2020), arXiv:2011.08647.

[71] A. Basu, "Proving relations between modular graph functions," Class. Quant. Grav. **33**, 235011 (2016), arXiv:1606.07084.

[72] A. Basu, "Low momentum expansion of one loop amplitudes in heterotic string theory," JHEP **11**, 139 (2017), arXiv:1708.08409.

[73] E. D'Hoker and J. Kaidi, "Modular graph functions and odd cuspidal functions. Fourier and Poincaré series," JHEP **04**, 136 (2019), arXiv:1902.04180.

[74] A. Basu, "Eigenvalue equation for the modular graph $C_{a,b,c,d}$," JHEP **07**, 126 (2019), arXiv:1906.02674.

[75] D. Dorigoni and A. Kleinschmidt, "Modular graph functions and asymptotic expansions of Poincaré series," Commun. Num. Theor. Phys. **13**, 569–617 (2019), arXiv:1903.09250.

[76] F. Brown, "A class of non-holomorphic modular forms I," Res. Math. Sci. **5**, 5:7 (2018), arXiv:1707.01230.

[77] F. Brown, "A class of non-holomorphic modular forms II : equivariant iterated Eisenstein integrals," Forum of Mathematics, Sigma **8**, 1 (2020), arXiv:1708.03354.

[78] D. Zagier and F. Zerbini, "Genus-zero and genus-one string amplitudes and special multiple zeta values," Commun. Num. Theor. Phys. **14**, 413–452 (2020), arXiv:1906.12339.

[79] J. Broedel, O. Schlotterer, and F. Zerbini, "From elliptic multiple zeta values to modular graph functions: open and closed strings at one loop," JHEP **01**, 155 (2019), arXiv:1803.00527.

[80] J.E. Gerken, A. Kleinschmidt, C.R. Mafra, O. Schlotterer, and B. Verbeek, "Towards closed strings as single-valued open strings at genus one," (2020), arXiv:2010.10558.

[81] O. Schnetz, "Graphical functions and single-valued multiple polylogarithms," Commun. Num. Theor. Phys. **08**, 589–675 (2014), arXiv:1302.6445.

[82] F. Brown and C. Dupont, "Single-valued integration and double copy," (2018), arXiv:1810.07682.

[83] A. Beilinson, A. Varchenko, A. Goncharov, and V. Shekhtman, "Projective Geometry and K-theory," Leningrad Math J. 2 **3**, 523–575 (1991).

[84] O. Schnetz, "Graphical hyperlogarithms," talk within the trimester "Periods in Number Theory, Algebraic Geometry and Physics", HIM, Bonn, Feb. 27, 2018.

[85] E. D'Hoker, M.B. Green, and B. Pioline, "Higher genus modular graph functions, string invariants, and their exact asymptotics," Commun. Math. Phys. **366**, 927–979 (2019), arXiv:1712.06135.

[86] E. D'Hoker and O. Schlotterer, "Identities among higher genus modular graph tensors," (2020), arXiv:2010.00924.

[87] B. Pioline, "A Theta lift representation for the Kawazumi-Zhang and Faltings invariants of genus-two Riemann surfaces," J. Number Theory **163**, 520 (2016), arXiv:1504.04182.

[88] R. Donagi and E. Witten, "Supermoduli Space Is Not Projected," Proc. Symp. Pure Math. **90**, 19–72 (2015), arXiv:1304.7798.

[89] A. Sen and E. Witten, "Filling the gaps with PCO's," JHEP **09**, 004 (2015), arXiv:1504.00609.

[90] E. Casali, S. Mizera, and P. Tourkine, "Monodromy relations from twisted homology," JHEP **12**, 087 (2019), arXiv:1910.08514.

[91] S.M. Chester, M.B. Green, S.S. Pufu, Y. Wang, and C. Wen, "Modular invariance in superstring theory from $\mathcal{N} = 4$ super-Yang-Mills," JHEP **11**, 016 (2020), arXiv:1912.13365.

[92] S.M. Chester, M.B. Green, S.S. Pufu, Y. Wang, and C. Wen, "New Modular Invariants in $\mathcal{N} = 4$ Super-Yang-Mills Theory," (2020), arXiv:2008.02713.

[93] R.R. Metsaev and A.A. Tseytlin, "Type IIB superstring action in AdS5 x S5 background," Nucl. Phys. B **533**, 109–126 (1998), arXiv:9805028.

[94] N. Berkovits and O. Chandia, "Superstring vertex operators in an AdS5 x S5 background," Nucl. Phys. B **596**, 185–196 (2001), arXiv:0009168.

[95] N. Berkovits, "Quantum consistency of the superstring in AdS(5) x S**5 background," JHEP **03**, 041 (2005), arXiv:hep-th/0411170.

[96] N. Berkovits and C. Vafa, "Towards a Worldsheet Derivation of the Maldacena Conjecture," AIP Conf. Proc. **1031**, 21–42 (2008), arXiv:0711.1799.

[97] D. Amati, M. Ciafaloni, and G. Veneziano, "Can Space-Time Be Probed Below the String Size?" Phys. Lett. B **216**, 41–47 (1989).

[98] N.E.J. Bjerrum-Bohr, J.F. Donoghue, and P. Vanhove, "On-shell Techniques and Universal Results in Quantum Gravity," JHEP **02**, 111 (2014), arXiv:1309.0804.