Perturbative Calculations of Anomalous Dimensions in Conformal Field Theories

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This Letter of Interest for the Snowmass 2021 Theory Frontier briefly reviews recent progress in perturbative calculations, up to the five-loop level, of anomalous dimensions of operators in a conformal field theory defined at an infrared fixed point in an asymptotically free gauge theory. Some examples of agreement between perturbative calculations and lattice gauge theory measurements and bootstrap results are given.

In a quantum field theory, the actual scaling dimension $D_{\mathcal{O}}$ of a generic operator $\mathcal{O}$ differs from its free-field value $D_{\mathcal{O},\rm free}$ due to interactions: $D_{\mathcal{O}} = D_{\mathcal{O},\rm free} - \gamma_{\mathcal{O}}$, where $\gamma_{\mathcal{O}}$ is the anomalous dimension of $\mathcal{O}$. Let us consider an asymptotically free vectorial non-Abelian gauge theory in four spacetime dimensions with gauge group G and N_f massless Dirac fermions transforming according to a representation R of G. The dependence of the gauge coupling $g = g(\mu)$ on the Euclidean momentum scale μ where it is measured is given by the beta function, $\beta = d\alpha/d\ln\mu$, where $\alpha = g^2/(4\pi)$. This has the series expansion

$$\beta = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell \ , \qquad (1)$$

where $a = \alpha/(4\pi)$. The asymptotic freedom condition implies $N_f < N_u$, where $N_u = 11C_A/(4T_f)$ [1]; C_A is the quadratic Casimir invariant of the group and $T_f = T(R)$ is the trace invariant of the fermion representation. There is an interval in N_f below N_u in which the theory flows from a weak-coupling limit in the deep ultraviolet (UV) to an infrared fixed point (IRFP) of the renormalization group (RG) at a nonzero value α_{IR}, where the beta function vanishes [2, 3], so that this limiting theory is scale invariant and is inferred to be conformally invariant. This interval is often called the conformal window or interval, where the IR theory is in a (deconfined) non-Abelian Coulomb phase with no spontaneous chiral symmetry breaking.

The values of the anomalous dimensions of (gauge-invariant) operators in the conformal field theory (CFT) at this IRFP are of fundamental interest. These include $\gamma_{\bar\psi\psi,IR}$, where the subscript indicates evaluation at the IRFP, and also $\beta'_{IR} = (d\beta/d\alpha)_{IR}$, which is (minus) the anomalous dimension of $F^a_{\mu\nu}F^{a\mu\nu}$, where $F^a_{\mu\nu}$ is the field strength tensor for the theory. Given the series expansion

$$\gamma_{\bar\psi\psi} = \sum_{\ell=1}^{\infty} c_\ell \, a^\ell \ , \qquad (2)$$

one can calculate an n-loop ($n\ell$) perturbative value of $\gamma_{\bar\psi\psi}$, denoted $\gamma_{\bar\psi\psi,n\ell}$, by first computing the value of the IR zero, denoted $\alpha_{IR,n\ell}$, of the n-loop beta function, and then substituting it in the n-loop expansion for $\gamma_{\bar\psi\psi}$. This was done to the four-loop level in [4, 5], to the five-loop level for SU(3) in [6], using b_5 from [7] and c_5 from [8], and to the five-loop level for general G and R in [9], using b_5 from [10]. The b_ℓ with $\ell \ge 3$ and c_ℓ with $\ell \ge 2$ are scheme-dependent; Refs. [7, 8, 10] used the $\overline{\rm MS}$ scheme [11]. The slope β'_{IR} was calculated to the four-loop level in [12, 13] and to five-loop level in [9]. In addition to results for general gauge group G and fermion representation R, explicit formulas were presented for $G = {\rm SU}(N_c)$ and R equal to the fundamental, adjoint, and rank-2 symmetric and antisymmetric tensor representations. These perturbative calculations are most accurate at the upper end of the conformal interval as a function of N_f, since $\alpha_{IR} \to 0$ as N_f approaches N_u from below. As N_f decreases, α_{IR} grows, and higher-loop terms become more important. Studies of scheme dependence were performed in [14].

It has been valuable to compare these perturbative calculations with lattice measurements of anomalous dimensions. For $G = {\rm SU}(3)$ and fermions in the fundamental (fund.) representation, the theory with $N_f = 12$ is considered by most lattice groups to be in the conformal interval [15–23] (see also [24, 25]). Our perturbative results for $\gamma_{\bar\psi\psi,IR,n\ell}$, calculated in the manner above, at n-loop levels $n = 3$ through $n = 5$, are (see Table I in [6]):

$$\gamma_{\bar\psi\psi,IR,3\ell} = 0.312, \quad \gamma_{\bar\psi\psi,IR,4\ell} = 0.253,$$
$$\gamma_{\bar\psi\psi,IR,5\ell} = 0.255 \ . \qquad (3)$$

Lattice measurements of $\gamma_{\bar\psi\psi,IR}$ for this theory include $\gamma_{\bar\psi\psi,IR} = 0.414(16)$ [16], $\gamma_{\bar\psi\psi,IR} \simeq 0.35$ [17], $\gamma_{\bar\psi\psi,IR} = 0.27(3)$ [19], $\gamma_{\bar\psi\psi,IR} \simeq 0.25$ [20], and $\gamma_{\bar\psi\psi,IR} = 0.235(46)$ [22]. Bootstrap methods provide another powerful approach to determine operator dimensions in CFTs [26, 27]. An application of these methods to this theory suggests a value $\gamma_{\bar\psi\psi,IR} \simeq 0.24$ [28]. Our four-loop and five-loop values from [6] are in good agreement with the lattice values from [19, 20, 22] and with the bootstrap value from [28]. For this theory, our value $\beta'_{IR,4\ell} = 0.282$ (see

Table IV in [12]), is in agreement with the lattice result $\beta'_{IR} = 0.26(2)$ [21].

Since $\alpha_{IR} \to 0$ as $N_f \to N_u$, one can reexpress anomalous dimensions at this type of IRFP as series in powers of the manifestly scheme-independent variable $\Delta_f = N_u - N_f$:

$$\gamma_{\bar\psi\psi,IR} = \sum_{n=1}^{\infty} \kappa_n (\Delta_f)^n \ . \tag{4}$$

The calculation of κ_n requires, as input, the b_ℓ with $1 \le \ell \le n+1$ and the c_ℓ with $1 \le \ell \le n$. This alternative method of calculating these anomalous dimensions has been carried out in [9, 13, 29–34]. To the maximal order to which $\gamma_{\bar\psi\psi,IR}$ has been calculated, namely $O(\Delta_f^4)$, the coefficients in this Δ_f series expansion are all positive. The results yield somewhat larger values for $\gamma_{\bar\psi\psi,IR}$ than the expansions in $\alpha_{IR,n\ell}$; for example, for the above-mentioned SU(3) theory with R = fund. and $N_f = 12$, $\gamma_{IR,\bar\psi\psi,IR,\Delta_f^3} = 0.323$ and $\gamma_{IR,\bar\psi\psi,IR,\Delta_f^4} = 0.338$ at $O(\Delta_f^3)$ and $O(\Delta_f^4)$, respectively (see Table I in [30]). Padé resummation methods yield similar values; the [1,2] and [0,3] Padé approximants to the $O(\Delta_f^4)$ series yield $\gamma_{\bar\psi\psi,IR,[1,2]} = 0.3375$ and $\gamma_{\bar\psi\psi,IR,[0,3]} = 0.352$, respectively (see Table II of Ref. [32]). Future work should further elucidate the predictions of these perturbative calculations of anomalous dimensions via series expansions in $\alpha_{IR,n\ell}$ and in Δ_f.

Other results include calculations of anomalous dimensions of baryon operators [33], of higher-spin fermion bilinear operators [34], and of operators in theories with fermions in two different representations [35]. After an initial study using series expansions in powers of $\alpha_{IR,n\ell}$ to calculate anomalous dimensions of operators in an $\mathcal{N} = 1$ supersymmetric gauge theory [36], these calculations were carried out using series expansions in powers of Δ_f in [37]. Importantly, the expansion coefficients calculated in [37] agree precisely, at each order, with the series expansion of the exactly known results for anomalous dimensions of operators in this theory [38, 39].

Although bilinear fermion operators are not, in general, gauge-invariant in chiral gauge theories, the $F^a_{\mu\nu}F^{a\mu\nu}$ operator is, and high-order perturbative calculations of β'_{IR} have also been carried out for the conformal interval of asymptotically free chiral gauge theories [40].

For the (non-supersymmetric) vectorial gauge theory, as N_f decreases toward the lower end of the conformal interval, α_{IR} increases, and as N_f decreases below a critical value denoted as $N_{f,cr}$, the IR behavior of the theory changes, with the onset of spontaneous chiral symmetry breaking and resultant dynamical mass generation, so that the IR theory is no longer a CFT. If N_f is only slightly less than $N_{f,cr}$, the theory may exhibit quasi-dilatation-invariant RG behavior with a slowly running gauge coupling. Such theories have been of interest for possible applications to models of physics beyond the Standard Model and have been extensively studied by lattice simulations. These have shown the appearance of a light scalar boson, consistent with being a dilaton-like state resulting from the spontaneous breaking of the approximate dilatation invariance [41–44]. Although our perturbative methods become less accurate as one approaches the lower end of the confirmal interval and α_{IR} gets larger, they can give a rough indication of where this strong-coupling behavior occurs and the corresponding value of $N_{f,cr}$. Our calculations suggest [30], for example, that for $G = \mathrm{SU}(3)$ and R = fund, $N_{f,cr} \sim 8-9$, in agreement with the lattice results in [23, 41–43].

We believe that this area is one where substantial progress has been made, both in higher-loop continuum perturbative calculations of anomalous dimensions of operators in conformal field theories and in fully nonperturbative lattice simulations. There have also been interesting connections with bootstrap CFT methods, and it is expected that more progress can be made in the near future. An example of a workshop where results from perturbative, lattice, and bootstrap CFT were discussed was [45]. Further interactions among these research communities should be fruitful.

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