Functions Beyond Polylogarithms in Scattering Amplitudes

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Some of the most essential and impactful work currently being done in amplitudes concerns the types of special functions that appear in scattering amplitudes beyond multiple polylogarithms. While multiple polylogarithms have long been known to appear in scattering amplitudes for small numbers of particles and at low loop orders (see for instance [1–3])—and indeed are sufficient for expressing one-loop amplitudes in any theory (in integer dimensions)—elliptic generalizations of polylogarithms are known to appear in QCD already at two loops [4]. In fact, it has recently been shown that integrals over even more complicated manifolds also appear in this (and other) theories, first at two loops where an integral over a K3 surface appears, and also at higher loops where integrals over Calabi-Yau manifolds with dimension proportional to the loop order contribute [5, 6]. While elliptic polylogarithms and the amplitudes containing them have received a great deal of attention in recent years [7–30], much remains to be understood about the algebraic and analytic properties of these types of functions. Amplitudes involving integrals over higher-dimensional manifolds remain even less well studied.

The potential benefits of better understanding these types of beyond-polylogarithmic functions is made clear by the 'polylog revolution' that has occurred in amplitudes over the last decade, as a result of our ever-increasing understanding of the geometric and analytic structure of multiple polylogarithms [31–41]. In particular, our ability to compute polylogarithmic amplitudes (via differential equations, direct integration, and bootstrap techniques) has taken huge leaps, for instance enabling us to compute the four-gluon amplitude in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory through three loops [42], six-particle amplitudes in the planar limit of this theory through seven loops [43], and certain infinite classes of integrals to all loop orders [44]. Many of these techniques can also be applied in QCD calculations, where promising progress has already been made (see for instance [45, 46]). This understanding of polylogarithms has also uncovered surprising combinatorial and number-theoretic structures embedded within scattering amplitudes [47–49], which connect the analytic structure of these amplitudes to cluster algebras, tropical geometry, and motivic Galois theory.

A similar revolution is needed in our understanding of the types of functions that appear in amplitudes beyond polylogarithms, both for making computational progress and because this promises to uncover yet deeper structures within quantum field theory. In particular, a better understanding of how to find and exploit identities between these types of functions is called for, as are efficient techniques for their numerical evaluation. While encouraging progress has been made on both of these fronts in the case of elliptic polylogarithms (see for instance [9,18,22,26,30]), the technology for dealing with elliptic polylogarithms remains far less developed than our technology for dealing with their non-elliptic counterparts.

Meanwhile, basic preparatory work is still needed to help delineate the types of integrals that appear in scattering amplitudes beyond elliptic polylogarithms. While the manifolds that have hitherto been identified in integrals contributing to generic gauge theories and scalar theories fall into a special subclass of Calabi-Yau manifolds [50,51], more refined analyses along the lines of [52] or using Picard-Fuchs-type differential equations (as in [50]) will help us better characterize these functions and their analytic properties.

Advances in these areas are required not just for understanding supersymmetric gauge theory and the formal structure of amplitudes, but also for making more precise predictions in the Standard Model. As such, these topics deserve to be a focal point of research over the next decade and more.

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