Solving Scattering in $\mathcal{N}=4$ Super-Yang-Mills Theory

Nima Arkani-Hamed,[1] Benjamin Basso,[2] Lance J. Dixon,[3] Andrew J. McLeod,[4] Marcus Spradlin,[5] Jaroslav Trnka,[6] and Anastasia Volovich[5]

[1] *School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, and Center of Mathematical Sciences and Applications, Harvard University, Cambridge, MA 02138*
[2] *Laboratoire de physique de l'Ecole normale supérieure, ENS, Université PSL, CNRS, Sorbonne Université, Université Paris-Diderot, Sorbonne Paris Cité, 24 rue Lhomond, 75005 Paris, France*
[3] *SLAC National Accelerator Laboratory, Stanford University, Stanford, CA 94309, USA*
[4] *Niels Bohr International Academy, Blegdamsvej 17, 2100 Copenhagen, Denmark*
[5] *Department of Physics, Brown University, Providence, RI 02912, USA*
[6] *Center for Quantum Mathematics and Physics (QMAP), Department of Physics, University of California, Davis, CA 95616, USA*

(Dated: September 3, 2020)

We highlight a continuing program of research into the structure of scattering amplitudes in $\mathcal{N}=4$ super-Yang-Mills theory, for its intrinsic understanding and for its relevance for collider physics.

The study of quantum scattering amplitudes has played a central role in the development of theoretical physics. Feynman diagrams represent the perturbative expansion of an amplitude as a sum over all ways the collisions between particles could take place in space-time. Feynman diagrams are where the rubber meets the road for quantum field theory: they underlie comparisons of theoretical predictions and experimental measurements at all high energy colliders, including the Large Hadron Collider (LHC), where quantum chromodynamics (QCD) plays the dominant role. Yet we now know, in part due to investigations of $\mathcal{N}=4$ super-Yang-Mills theory ($\mathcal{N}$=4 SYM), that there are completely different ways of visualizing scattering amplitudes, where pictures of particle trajectories, and indeed the notion of space-time itself, make no direct appearance.

Beyond the very simplest processes, direct Feynman diagram calculations can be incredibly complicated. Yet amazingly, the final results, obtained by summing pages of algebra, often collapse to single-term expressions! The more we have learned to calculate scattering amplitudes, especially in $\mathcal{N}$=4 SYM, the more we have uncovered seemingly miraculous patterns, simplicity and symmetries, reflecting beautiful new mathematical structures, which are at the center of modern mathematics research, including combinatorics, positive Grassmannians, cluster algebras, number theory, and the theory of motives.

In the planar limit of a large number of colors, scattering in $\mathcal{N}$=4 SYM is a laboratory for addressing analytically the dynamical properties of quantum field theory in Minkowski spacetime. Today we have *three* independent descriptions of scattering amplitudes in planar $\mathcal{N}$=4 SYM. A weak-coupling formulation makes contact with standard Feynman diagrammatic perturbation theory [1, 2]; a "holographic" strong-coupling formulation employs minimal area surfaces in Anti-de Sitter space [3, 4]; and the Pentagon Operator Product Expansion (OPE) approach exploits the two-dimensional integrability of a dual string picture at finite coupling in various kinematic limits [5–7]. The three formulations are all mutually consistent, but they explore different physics, use seemingly different mathematics, and make different properties of amplitudes manifest. The task for the future is to answer the physical question: *How do all these pieces of the puzzle fit in within a single unifying description of scattering amplitudes and quantum field theory?* There is a corresponding mathematical question: what are the right functions to express all facets of the answer, from weak to strong coupling and for arbitrary kinematics?

Facets of these overarching questions include: How do gluonic and stringy descriptions morph into each other as the coupling and kinematics are varied? What kind of singularities show up and what is the physics associated to them? How do holographic dualities, string theory, and even space-time itself, emerge dynamically from planar gauge theories? Solving scattering in planar $\mathcal{N}$=4 SYM will provide a quantitative test for physical and mathematical expectations, and will lead to improved intuition for the behavior of more general theories.

Returning to perturbation theory, Feynman diagrams express loop amplitudes as a sum over terms, each of which must be integrated over the loop momenta. For gauge theories in the planar limit of a large number of colors, these terms can be combined into a single object, the loop integrand or *scattering form* [8]. Scattering forms for massless theories can be computed by "on-shell diagrams", which arise from gluing three-particle interactions together, not at points in space-time as with Feynman diagrams, but non-locally along the lightcones associated with massless particle trajectories [9]. Individual on-shell diagrams in any planar theory are associated with a certain differential form on a manifold called the positive Grassmannian.

The scattering form for massless planar $\mathcal{N}$=4 SYM is particularly simple, and it can be determined recursively by exploiting the physical requirements of locality and unitarity. Even more remarkably, it can be defined as "the volume" of the *amplituhedron* [10, 11], a double generalization of the positive Grassmannian, roughly analogous to the way convex plane polygons generalize triangles. The scattering form is the unique form with logarithmic singularities on all boundaries of the amplituhedron. In this approach, scattering amplitudes are uniquely dictated by geometry. The basic rules of spacetime and quantum mechanics can therefore arise as *outputs* of geometry, rather than as fundamental starting points. The exploration of these geometric aspects of particle scattering is still in its infancy. A full determination of the topology and cell structure of the amplituhedron is a clear goal for the coming years, as is the nonperturbative generalization of these notions.

Given a scattering form, one must perform the loop integrals to obtain an amplitude. In many cases, in both QCD and $\mathcal{N}=4$ SYM, the integrals are generalized polylogarithms in the Mandelstam variables (or certain dual conformal cross ratios formed from them) [12–18]. More complicated functions also appear: elliptic polylogarithms, associated with genus 1 curves [19–25], and "beyond elliptic" functions associated with higher dimensional Calabi-Yau manifolds [26, 27]. These transcendental functions have strong physical restrictions on their branch cuts: They can only appear (on the first sheet) at physical thresholds, and double discontinuities in partially overlapping channels are forbidden by the Steinmann relations [28, 29]. In the case of planar $\mathcal{N}=4$ SYM, these constraints, along with a knowledge of certain other singularities, have allowed six-point amplitudes to be bootstrapped through seven loops [30, 31] (and seven-point amplitudes through four loops [32–34]), by simply writing the answer as a linear combination of the right functions and then imposing physical constraints, without any direct reference to the underlying loop integrand. A future challenge is to extend these methods to the elliptic case and beyond, for applications to $\mathcal{N}=4$ SYM and to QCD for the LHC. For the LHC, a premium will be placed on efficient numerical evaluation in all regions of the physical phase space, but such efficiency may benefit from a bootstrapping perspective as well.

Another very exciting new direction is the connection between scattering amplitudes and cluster algebras [35, 36]. One connection is at the level of the scattering form: the positive Grassmannian is naturally endowed with a cluster structure [9]. Another connection is to the structure of multi-loop amplitudes after integrating over all the loop momenta [2, 37–39]. In the best understood cases, the singularities of the polylogarithmic functions are located at the vanishing loci of cluster coordinates on the Grassmannian cluster algebra $\mathrm{Gr}(4, n)$. The appearance and organization of cluster variables in amplitudes in multiple ways strongly suggests that scattering amplitudes should be thought of as answers to deep new mathematical questions rooted in cluster geometry.

It has recently become clear that surprisingly many of the remarkable properties of planar $\mathcal{N}=4$ SYM generalize to the nonplanar sector [40], and even to (super)gravity [41, 42], where there is no separation between planar and nonplanar. Still to be identified are the mathematical structures that play the same role as the amplituhedron does in the planar case. An improved understanding of scattering forms for nonplanar $\mathcal{N}=4$ SYM could be extended to quantum (super)gravities and thereby help to determine their ultraviolet behavior.

Our understanding of $\mathcal{N}=4$ SYM beyond the planar limit has already contributed to results relevant for collider physics, in the area of infrared resummation and other soft-gluon effects. At higher perturbative orders, soft effects can connect more hard partons together. The first time they are connected by a web of gluons, the contribution is the same in $\mathcal{N}=4$ SYM as in QCD. This feature was exploited for the terms in the three-loop soft anomalous dimension connecting four hard partons [43, 44], and more recently in the computation of the two-loop amplitude for soft-gluon emission in the presence of three hard partons [45]. More results of a similar variety will surely appear in the future.

We believe that the continued intensive study of scattering in $\mathcal{N} = 4$ SYM will further transform our understanding of quantum scattering, with multiple payoffs for high-energy physics that are both conceptual and practical in nature.

[1] R. Feynman, Phys. Rev. **76**, 769 (1949).
[2] S. Caron-Huot, L. J. Dixon, J. M. Drummond, F. Dulat, J. Foster, Ö. Gürdoğan, M. von Hippel, A. J. McLeod, and G. Papathanasiou, PoS **CORFU2019**, 003 (2020), arXiv:2005.06735 [hep-th].
[3] L. F. Alday and J. M. Maldacena, JHEP **06**, 064 (2007), arXiv:0705.0303 [hep-th].
[4] L. F. Alday, J. Maldacena, A. Sever, and P. Vieira, J. Phys. A **43**, 485401 (2010), arXiv:1002.2459 [hep-th].
[5] L. F. Alday, D. Gaiotto, J. Maldacena, A. Sever, and P. Vieira, JHEP **04**, 088 (2011), arXiv:1006.2788 [hep-th].
[6] B. Basso, A. Sever, and P. Vieira, Phys. Rev. Lett. **111**, 091602 (2013), arXiv:1303.1396 [hep-th].
[7] B. Basso, S. Caron-Huot, and A. Sever, JHEP **01**, 027 (2015), arXiv:1407.3766 [hep-th].
[8] N. Arkani-Hamed, Y. Bai, S. He, and G. Yan, JHEP **05**, 096 (2018), arXiv:1711.09102 [hep-th].
[9] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, A. B. Goncharov, A. Postnikov, *et al.*, (2012), arXiv:1212.5605 [hep-th].
[10] N. Arkani-Hamed and J. Trnka, JHEP **10**, 030 (2014), arXiv:1312.2007 [hep-th].
[11] N. Arkani-Hamed and J. Trnka, JHEP **12**, 182 (2014), arXiv:1312.7878 [hep-th].
[12] K.-T. Chen, Bull. Amer. Math. Soc. **83**, 831 (1977).
[13] A. Goncharov, Adv. Math. **114**, 197 (1995).

[14] E. Remiddi and J. Vermaseren, Int.J.Mod.Phys. **A15**, 725 (2000), arXiv:hep-ph/9905237 [hep-ph].
[15] A. Goncharov, (2001), arXiv:math/0103059 [math.AG].
[16] F. C. Brown, C. R. Acad. Sci. Paris, Ser. I **338**, 527 (2004).
[17] A. B. Goncharov, M. Spradlin, C. Vergu, and A. Volovich, Phys.Rev.Lett. **105**, 151605 (2010), arXiv:1006.5703 [hep-th].
[18] C. Duhr, JHEP **1208**, 043 (2012), arXiv:1203.0454 [hep-ph].
[19] A. Sabry, Nucl. Phys. **33**, 401 (1962).
[20] D. J. Broadhurst, Z. Phys. C **47**, 115 (1990).
[21] S. Caron-Huot and K. J. Larsen, JHEP **10**, 026 (2012), arXiv:1205.0801 [hep-ph].
[22] S. Bloch, M. Kerr, and P. Vanhove, Adv. Theor. Math. Phys. **21**, 1373 (2017), arXiv:1601.08181 [hep-th].
[23] E. Remiddi and L. Tancredi, Nucl. Phys. B **925**, 212 (2017), arXiv:1709.03622 [hep-ph].
[24] J. L. Bourjaily, A. J. McLeod, M. Spradlin, M. von Hippel, and M. Wilhelm, Phys. Rev. Lett. **120**, 121603 (2018), arXiv:1712.02785 [hep-th].
[25] J. Broedel, C. Duhr, F. Dulat, and L. Tancredi, JHEP **05**, 093 (2018), arXiv:1712.07089 [hep-th].
[26] S. Bloch, M. Kerr, and P. Vanhove, Compos. Math. **151**, 2329 (2015), arXiv:1406.2664 [hep-th].
[27] J. L. Bourjaily, Y.-H. He, A. J. Mcleod, M. Von Hippel, and M. Wilhelm, Phys. Rev. Lett. **121**, 071603 (2018), arXiv:1805.09326 [hep-th].
[28] O. Steinmann, Helv. Physica Acta **33**, 257 (1960).
[29] O. Steinmann, Helv. Physica Acta **33**, 347 (1960).
[30] S. Caron-Huot, L. J. Dixon, A. McLeod, and M. von Hippel, Phys. Rev. Lett. **117**, 241601 (2016), arXiv:1609.00669 [hep-th].
[31] S. Caron-Huot, L. J. Dixon, F. Dulat, M. von Hippel, A. J. McLeod, and G. Papathanasiou, JHEP **08**, 016 (2019), arXiv:1903.10890 [hep-th].
[32] L. J. Dixon, J. Drummond, T. Harrington, A. J. McLeod, G. Papathanasiou, and M. Spradlin, JHEP **02**, 137 (2017), arXiv:1612.08976 [hep-th].
[33] J. Drummond, J. Foster, Ö. Gürdoğan, and G. Papathanasiou, JHEP **03**, 087 (2019), arXiv:1812.04640 [hep-th].
[34] L. J. Dixon and Y.-T. Liu, (2020), arXiv:2007.12966 [hep-th].
[35] S. Fomin and A. Zelevinsky, (2001), arXiv:math/0104151 [math.RT].
[36] S. Fomin and A. Zelevinsky, (2002), arXiv:math/0208229 [math.RA].
[37] J. Golden, A. B. Goncharov, M. Spradlin, C. Vergu, and A. Volovich, JHEP **1401**, 091 (2014), arXiv:1305.1617 [hep-th].
[38] J. Drummond, J. Foster, and Ö. Gürdoğan, Phys. Rev. Lett. **120**, 161601 (2018), arXiv:1710.10953 [hep-th].
[39] J. Golden, A. J. McLeod, M. Spradlin, and A. Volovich, JHEP **03**, 195 (2019), arXiv:1902.11286 [hep-th].
[40] Z. Bern, E. Herrmann, S. Litsey, J. Stankowicz, and J. Trnka, JHEP **06**, 098 (2016), arXiv:1512.08591 [hep-th].
[41] E. Herrmann and J. Trnka, JHEP **02**, 084 (2019), arXiv:1808.10446 [hep-th].
[42] A. Edison, E. Herrmann, J. Parra-Martinez, and J. Trnka, (2019), arXiv:1909.02003 [hep-th].
[43] Ø. Almelid, C. Duhr, and E. Gardi, Phys. Rev. Lett. **117**, 172002 (2016), arXiv:1507.00047 [hep-ph].
[44] Ø. Almelid, C. Duhr, E. Gardi, A. McLeod, and C. D. White, JHEP **09**, 073 (2017), arXiv:1706.10162 [hep-ph].
[45] L. J. Dixon, E. Herrmann, K. Yan, and H. X. Zhu, JHEP **05**, 135 (2020), arXiv:1912.09370 [hep-ph].