On The Need For Path Integral Contour Deformations To Tame the Sign Problem

Neill C. Warrington[1]

[1] *Institute for Nuclear Theory, University of Washington, Seattle, WA 98195-1550*[*]

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The sign problem appears in many problems of interest in lattice gauge theory, spanning from finite density QCD to all real time phenomena, and currently prevents their solution. Due to this wide range of problems affected, generic tools to reduce the sign problem are useful. A unique approach to tame the sign problem has emerged in the past several years (for a review see [1]); the purpose of this Letter is to argue for its usefulness and the need for its further development by the community.

The approach discussed here stems from the observation that the path integral, when discretized on a lattice, is a finite-dimensional integral. As such, it is amenable to all the theorems of multi-dimensional complex analysis; this includes the multi-dimensional generalization of Cauchy's integral theorem, which states that the integral of a holomorphic function remains unchanged when the manifold of integration is deformed, provided non-analytic points are not crossed [1]. This theorem implies much freedom in the choice integration domain of the path integral, and this freedom may be used to tame the sign problem.

Several methods for deforming the path integral have been developed and have been applied with success to hard problems. The holomorphic gradient flow, which deforms the integration manifold smoothly by a particular differential equation, has been used to tame the sign problem in a variety of systems, including notably $(1 + 1)$ dimensional real-time problems [2]. The sign-optimized manifold method, which minimizes the sign problem via gradient ascent, has rendered tractable the calculation of the phase diagram of a (2+1)D fermionic model resembling QCD. Manifold deformations can be used to tame more than just sign problems, too. They can be used to tame signal-to-noise problems, which plague, for example, the calculation of the baryon mass from lattice QCD. This has been demonstrated recently in two low dimensional toy models [3].

Theoretical developments in manifold deformation techniques are needed before very challenging real-world systems, such as finite density QCD, can be solved, however. The Jacobian associated with the holomorphic gradient flow is expensive to compute and must be sped-up. More robust sign-optimized manifolds are needed to tackle higher dimensional theories. Useful manifolds for non-abelian gauge theories are today nearly non-existent and must be found.

While there is much to be done, this effort would be well-invested and for several reasons. First, the examples cited above provide evidence that manifold deformations may tame some real-world sign problems; we cannot solve finite density QCD without trying. Second, there is currently no other technique to compute the real-time dynamics of a non-perturbative system with systematically controllable errors which can be extrapolated to zero. This alone makes this technique worth exploration and development. However, given as well the emergence of quantum computing as a technique to compute real-time observables, it is all the more important to have classically-obtained results with which to compare quantum calculations.

[1] A. Alexandru, G. Basar, P. F. Bedaque, and N. C. Warrington, "Complex paths around the sign problem," (2020), arXiv:2007.05436 [hep-lat].
[2] A. Alexandru, G. Başar, P. F. Bedaque, and G. Ridgway, Physical Review D **95** (2017), 10.1103/physrevd.95.114501.
[3] W. Detmold, G. Kanwar, M. L. Wagman, and N. C. Warrington, Physical Review D **102** (2020), 10.1103/physrevd.102.014514.

* ncwarrin@gmail.com

[1] A holomorphic function is the multi-dimensional generalization of an analytic function.