Perturbative calculations: elliptic contributions

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Abstract

This letter of interest for the Theory Frontier of the Snowmass planning exercise focuses on transcendental functions related to elliptic curves, which appear in precision calculations in perturbative quantum field theory from two-loop onwards.

Extended outline

The Standard Model involves several heavy particles: the Z- and W-bosons, the Higgs boson and the top quark. Precision studies of these particles require on the theoretical side quantum corrections at the two-loop order and beyond. It is a well-known fact that starting from two-loops Feynman integrals can no longer be expressed exclusively in terms of multiple polylogarithms. Transcendental functions beyond multiple polylogarithms appear quite early on at two-loops, as soon as massive particles are involved.

Multiple polylogarithms, either defined by their sum representation

$$\text{Li}_{m_1,\dots,m_k}(x_1,\dots,x_k) = \sum_{n_1>n_2>\dots>n_k>0}^{\infty} \frac{x_1^{n_1}}{{n_1}^{m_1}} \dots \frac{x_k^{n_k}}{{n_k}^{m_k}} \quad (1)$$

or their integral representation

$$G(z_1,\dots,z_k;y) = \int\limits_0^y \frac{dy_1}{y_1-z_1} \int\limits_0^{y_1} \frac{dy_2}{y_2-z_2} \dots \int\limits_0^{y_{k-1}} \frac{dy_k}{y_k-z_k} \quad (2)$$

are associated to a Riemann surface of genus zero (the Riemann sphere).

Starting from two-loops we also encounter transcendental functions associated to a Riemann surface of genus one (an elliptic curve). These are dubbed “elliptic multiple polylogarithms” and the focus of this contribution. Let me outline the three main points, which will be discussed in the white paper:

1. Various definitions of "elliptic multiple polylogarithms" exist in the literature. These definitions differ in details. The white paper will compare the various definitions. While some differences in the definition are just a matter of conventions, let me highlight a more fundamental difference: Some authors prefer a definition, where the functions are double-periodic at the expense of not being meromorphic. Other authors prefer a definition, where the functions are meromorphic, at the expense of not being double-periodic. It is not possible to have both. For the application towards Feynman integrals, the latter definition is appropriate. Thus we are not dealing with a single-valued function on an elliptic curve, but either a multi-valued function on an elliptic curve or a single-valued function on a covering space of an elliptic curve.

2. The standard method for computing Feynman integrals is the method of differential equations. We would like that the result for a Feynman integral follows easily from the differential equation, once the differential equation is brought into a canonical form. Thus we adopt a definition of elliptic multiple polylogarithms in terms of iterated integrals. The relevant space for these iterated integrals is the moduli space of a Riemann surface of genus one with n marked points. Standard coordinates on this space are $(n-1)$-marked points (one point may be fixed by translation invariance at the origin) and the modular parameter τ. Thus we may either integrate the differential equation in a z-variable or in τ, giving two alternatives how to express the Feynman integral. The white paper will contrast these two possibilities.

3. At the end of the day we would like to evaluate the resulting transcendental functions numerically. The standard method for the numerical evaluation is a converging sum representation, which can be truncated after a sufficient precision has been reached. The numerical evaluation of transcendental functions related to elliptic curves is very often based on q-expansions (with $q = \exp(2\pi i\tau)$) and the white paper will discuss numerical evaluations.