The Elimination of Renormalization Scale Ambiguities in pQCD

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Precise pQCD predictions for physical observable play a crucial role in testing the Standard Model (SM) and the identification of new physics beyond the SM.

It has been conventional to guess the renormalization scale μ_r to represent the characteristic momentum flow Q of a process, or to minimize large logarithmic corrections. The uncertainties for predictions based on a guessed scale are usually estimated by varying it over an arbitrary range; e.g., $\mu_r \in [Q/2, 2Q]$. The principle of *renormalization group invariance (RGI)* is the principle that a physical observable must be independent of the choices of both the renormalization scale and scheme. However, guessing the choice of μ_r violates the RGI and introduces arbitrary scheme-and-scale dependences in pQCD predictions.

Guessing the scale is also inconsistent with the well-known Gell-Mann-Low (GM-L) procedure used in QED [1]. The GM-L procedure determines the correct renormalization scale by setting the scale to the virtuality of the exchanged photons; the vacuum polarization diagrams (the QED $\{\beta_i\}$-terms) are eliminated, since they are summed into the QED running coupling. There is thus no ambiguity in setting the renormalization scale in QED.

Conventional scale-setting also has the negative consequence that the resulting pQCD series suffers from a divergent *renormalon* ($\alpha_s^n \beta_0^n n!$) series [2] characteristic of a nonconformal series at order n, where α_s is the running QCD coupling. Furthermore, the theoretical error estimated by simply varying μ_r over an arbitrary range is clearly unreliable, since it only partly reflects the unknown perturbative contributions from the nonconformal terms.

The Principle of Maximum Conformality (PMC) [3–7] provides a systematic way to eliminate the renormalization scheme-and-scale ambiguities. It has a rigorous theoretical foundation, satisfying RGI [8, 9] and all of the self-consistency conditions derived from the renormalization group [10]. The PMC scales at each order are obtained by shifting the argument of α_s to eliminate all the non-conformal $\{\beta_i\}$-terms; the resulting perturbative series thus matches the conformal series with $\beta = 0$; the PMC scales thus reflect the virtuality of the propagating gluons for the QCD processes. The divergent renormalon contributions are eliminated since they are summed in α_s. The PMC reduces in the $N_C \to 0$ Abelian limit [11] to the GM-L method. The resulting PMC scales also determine the correct effective numbers of active flavors n_f at each order. Moreover, the pQCD convergence is automatically improved due to the elimination of the divergent renormalon series.

A crucial point is that the resulting scale-fixed predictions for physical observables using the PMC are *independent of the choice of renormalization scheme* such as $\overline{MS}$ – a key requirement of RGI. The PMC predictions are also independent of the choice of the *initial* renormalization scale μ_r.

The PMC is also the theoretical principle underlying the Brodsky-Lepage-Mackenzie (BLM) procedure [12], *commensurate scale relations* connecting observables, as well as the scale-setting method used in lattice gauge theory. One can also use an additional property of renormalizable SU(N)/U(1) gauge theories, "Intrinsic Conformality (iCF)", which underlies the scale invariance of physical observables. This method, PMC_∞ [13] leads to a remarkably efficient method for eliminating the renormalization scale ambiguity *at every order* in pQCD.

The scale dependence of α_s is controlled by the renormalization group equation, which can be used recursively to establish the perturbative pattern of $\{\beta_i\}$-terms at each order [14]. The pQCD prediction (ϱ) for a physical observable can be expressed as [6, 7]

$$\begin{aligned}\varrho(Q) \;=\; & r_{0,0} + r_{1,0}\alpha(\mu_r) + \left[r_{2,0} + \beta_0 r_{2,1}\right]\alpha^2(\mu_r) + \\ & \left[r_{3,0} + \beta_1 r_{2,1} + 2\beta_0 r_{3,1} + \beta_0^2 r_{3,2}\right]\alpha^3(\mu_r) + \cdots \quad (1)\end{aligned}$$

where $\alpha = \alpha_s/4\pi$, and Q is the scale at which the observable is measured. All the coefficients $r_{i,j}$ are, in principle, functions of the initial choice of scale μ_r and Q, in which the coefficients $r_{i,0}$ are conformal parts of the coefficients. After applying the standard PMC procedure, the final pQCD prediction for ϱ reads

$$\varrho(Q) = r_{0,0} + \sum_{i=1}^{n} r_{i,0}\alpha^i(Q_i), \quad (2)$$

where Q_i are PMC scales, which are generally free of μ_r-dependence and can be interpreted as the relevant "physical" scales of the perturbative graphs at each of order i contributing to the physical observable. PMC predictions can be made for observables with multiple scales. The residual scale dependence due to unknown higher-order perturbative terms is highly suppressed [9]. The coefficients of the resulting series Eq. (2) match the coefficients of the corresponding conformal theory with $\beta = 0$. As in QED, all $\{\beta_i\}$-terms are absorbed into the scale of α_s.

The PMC approach has now been successfully applied to many different high energy processes, including Higgs boson production at the LHC [15], Higgs boson decays to $\gamma\gamma$ [16, 17], gg and $b\bar{b}$ [18–20] processes, top-quark pair production at the LHC and Tevatron [4, 21–25], semihard processes based on the BFKL approach [26–29], electron-positron annihilation to hadrons [6–8], the hadronic Z^0 boson decays [30, 31], event shapes in electron-positron annihilation [13, 32, 33], $\Upsilon(1S)$ leptonic decay [34, 35], charmonium production [36–38], and various decay processes [39, 40]. In addition, the PMC provides a possible solution to the $B \to \pi\pi$ puzzle [41] and the $\gamma\gamma^* \to \eta_c$ puzzle [42].

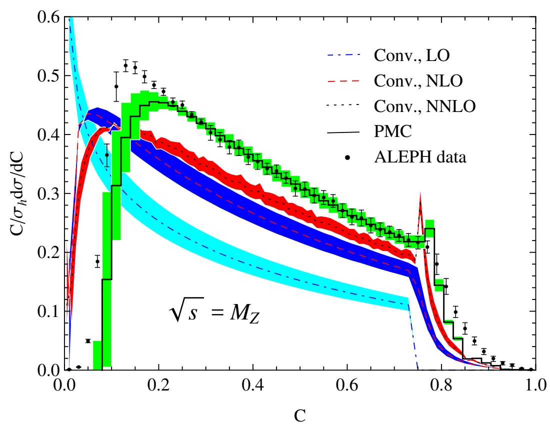

FIG. 1: The C-parameter differential distributions using conventional (Conv.) and PMC scale settings at $\sqrt{s} = M_Z$. The dot-dashed, dashed and dotted lines are the conventional scale-fixed results at LO, NLO and NNLO [43, 44], respectively. The solid line is the PMC result. The data are taken from the ALEPH [45] experiment.

As an explicit example of the PMC, it is found that by using the PMC, one has an elegant way to determine the running of α_s from the event shape distributions measured via the electron-positron annihilation [33]. In the case of conventional scale-setting, one simply sets the renormalization scale to be the center-of-mass collision energy $\mu_r = \sqrt{s}$, and the event shape distributions do not match the precise experimental data. Worse, only one value of α_s at the scale $\sqrt{s}$ can be extracted, whose main error source is the choice of the renormalization scale μ_r. On the other hand, the PMC scale can be determined with the help of the known $\{\beta_i\}$ terms, which monotonically increases with the value of event shapes, well reflecting the increasing virtuality of the QCD dynamics. More explicitly, we present the typical C-parameter using the conventional and PMC scale-settings in Fig. 1. Figure 1 shows that the conventional predictions – even up to NNLO pQCD corrections – substantially deviate

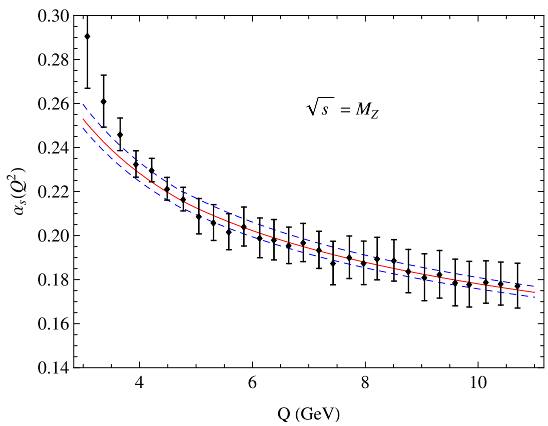

FIG. 2: The coupling constant $\alpha_s(Q^2)$ extracted by comparing PMC predictions with the ALEPH data [45] at a single energy of $\sqrt{s} = M_Z$ from the C-parameter distributions in the $\overline{\text{MS}}$ scheme. The error bars are the squared averages of the experimental and theoretical errors. The three lines are the world average evaluated from $\alpha_s(M_Z^2) = 0.1181 \pm 0.0011$ [46].

from the precise experimental data; the estimate of unknown higher-order terms by varying $\mu_r \in [\sqrt{s}/2, 2\sqrt{s}]$ is unreliable, In addition, the perturbative series for the C-parameter distribution shows slow convergence because of the renormalon problem. In contrast the PMC prediction for the C-parameter distribution is in excellent agreement with the experimental data.

Since the PMC scale in $\alpha_s(Q^2)$ varies with the value of the event shape C, we can extract $\alpha_s(Q^2)$ over a wide range of Q^2 using the experimental data at a single energy of $\sqrt{s}$, as shown in Fig 2. The results for $\alpha_s(Q^2)$ in the range 3 GeV $< Q <$ 11 GeV are in excellent agreement with the world average evaluated from $\alpha_s(M_Z^2)$ [46]. Thus, as required, the PMC approach eliminates the renormalization scale uncertainty, and the extracted $\alpha_s(Q^2)$ is not plagued by any uncertainty from the choice of the renormalization scale μ_r. One can also use the PMC to improve α_s near the nonperturbative domain. Knowing α_s at low momentum is often critical even for perturbative studies, since some of the physical scales Q_i may be soft. In fact, following the GM-L prescription for couplings makes α_s an observable; i.e., an *effective charge*. Thus, the PMC procedure is applicable to α_s itself. The resulting PMC series for α_s [47] does not exhibit the severe renormalon growth conspicuous in the standard α_s pQCD series [14]. This leads to significantly smaller uncertainties in the behavior of α_s, especially at lower Q values.

The applications of the PMC illustrate the importance of correct, rigorous renormalization scale-setting. In virtually every application, it is found that the application of the PMC systematically eliminates a major theoretical uncertainty for pQCD predictions; the application of the PMC thus greatly increases collider sensitivity to possible new physics beyond the Standard Model.

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