Comments on Collinear Factorization

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Abstract

In this short review, I'll discuss on the background, applicability and prospects of collinear factorization.

Contents

1 Introduction: the Concepts of Factorization

Factorization is the fulcrum of the collider system, which is designed to lift new worlds from the vacuum, with high-energy colliding protons providing the lever arm. It is central to the energy frontier program, entering into nearly every attempt to achieve precision within the Standard Model, and in recognizing signals of new physics [1, 2]. As our searches become more sophisticated, they become more sensitive to dynamics of the final state. As we shall recall, such sensitivity may violate the assumptions necessary to prove factorization. The purpose of this little essay is to present one person's viewpoint on the basic concepts that have gone into proofs of collinear factorization at hadron colliders, the fundamental results, and some of their basic strengths and limitations, to suggest where we can be confident, and where more work remains to be done. The presentation is a mixture of intuitive and technical arguments, hopefully in a conversational style. It is thus by no means intended to provide a final word, but rather to lead to further conversation as the field moves forward into the next era of collider physics.

Factorization is the quantum field theoretic realization of the parton model, in which large momentum transfer cross sections in hadron-hadron (or lepton-hadron) collisions are the result of the short-distance collisions of constituents within those hadrons, each Lorentz-contracted and time dilated in the frame of the other. The word "factorization" in this context refers to our ability to treat the momentum distributions of the partons as incoherent from the probability of their short-distance interactions, if and only to the extent that they collide at relative velocities approaching the speed of light. At the moment of collision, each hadron is in a superposition of quantum states that is completely independent of the other hadron. At high enough energies, all hadrons are "pristine" at the moment of collision, in the same mixture of states since baryogenesis. This implies universality of the kinematic distributions of constituents in fractional momenta, the parton distributions. In collinear factorization, the momenta of the colliding partons are lightlike and in the directions of the colliding hadrons. The basic concept here is *causality*, in the sense that signals from one hadron cannot reach the other before the moment of collision.

The collision takes place over a short time scale, creating new particles and/or patterns of momentum flow that evolve into the observed final state. Interactions that take place long after the short-distance process involve smaller momentum transfers, which for suitably-defined observables do not alter the overall flow of momentum, or excite quantum numbers in a manner that can neutralize those created the hard scattering. When summed with sufficient inclusivity, perturbative singularities associated with this evolution cancel, because they do not affect the total probability that defines the set of states that are summed. This is an expression of the *unitarity* of the underlying theory. Here, the identifiable class of final states is typically characterized by decay products of transient, heavy degrees of freedom and/or high-p_T leptons or jets. These may be Standard Model or not, and final states may be visible or dark, so long as we can recognize their presence, directly or indirectly.

Along with causality and unitarity, another central concept of factorization is *universality*. Most commonly, this refers to parton distributions, but in a closely related sense universality also applies to hard scattering functions. For parton distributions, it applies between different processes, say the proton distributions in deep-inelastic scattering for an electron-proton initial state, and in Drell-Yan, or jet production production for hadron-hadron scattering. For the short-distance hard-scattering factors, we assume a universality between perturbative calculations with different long-distance completions of the theory. For example, the production of a photon plus gluon from a quark pair is the same in real QCD as in the four-dimensional limit of dimensionally-continued QCD. The universality of parton distributions allows us to transfer information from deep-inelastic scattering to hadron-hadron scattering and vice-versa. The universality of short-distance factors allows us to compute hard scattering for incoming partons rather than model hadrons, clearly a much more efficient procedure.

In principle, as we go to higher orders in hard scatterings, simultaneously fitting more precisely determined parton distributions within processes we confidently identify as Standard Model at short distances, we can achieve more precision in the comparison of high-statistics data to our calculations, with an eye out for discrepancies. At the same time,

factorization allows us to calculate with the same precision the effects of new forces and fields to these same data. In the best of possible worlds, we may see behavior that is qualitatively different from the Standard Model predictions (asymmetries, etc.) In all this, uncertainties in data, and in parton distributions are subjects of increasing interest.

Collinear factorization is well-illustrated by the inclusive cross section for the production of a Drell-Yan pair or more generally, a heavy electroweak boson of mass Q,

$$Q^4 \frac{d^2\sigma^{pp\to\gamma^*(Q)X}}{dQ^2} = \sum_{ab} \int dx_a\, f_{a/p}(x_a) \int dx_b\, f_{b/p}(x_b)\, \omega^{ab\to\gamma^*(Q)}(x_a P_A, x_b P_B, Q)\,, \qquad (1)$$

with the $f_{i/p}$ parton distributions, and $\omega^{ab\to\gamma^*(Q)}(x_a P_A, x_b P_B, q)$ the perturbatively calculable short-distance functions for partons a and b to produce an off-shell photon of mass Q. Similarly, jet cross sections take the generic collinear-factorized form

$$S\,\sigma^{pp\to\{j_i\}X} = \sum_{ab} \int dx_a\, f_a(x_a) \int dx_b\, f_b(x_b)\, \omega^{ab\to\{c_i(p_i)\}X}(x_a P_A, x_b P_B, \{p_i, \delta_i\})\,. \qquad (2)$$

where $\{p_i, \delta_i\}$ labels the momenta of the observed jets and the paramters δ_i that define them, jet cone radii, for example. In both cases, the short-distance, perturbative function, ω, is normalized to be dimensionless. We have suppressed the factorization scale, which for this discussion we assume is of the order of the scale of renormalization. Clear generalizations exist to multiple electroweak boson production followed by decay, and to an unlimited set of "new physics" scenarios, as long as these involve heavy particles and short-distance interactions.

2 What's involved in factorization proofs

As suggested above, the basic ideas that go into arguments for factorization are causality and unitarity. In informal language, the formation of initial states takes place too early to affect the hard scattering, and subsequent interactions take place to late to undo it. Also as hinted above, the applicability of these ideas very much depends on what we measure in the final states, and how we combine them.

First a word about what is meant by *proof* in this context. All proofs I know of assume that hard-scattering functions are universal, and that they can be identified in the scattering of elementary quanta, generally in a theory that is infrared-regulated dimensionally or by other means. In that theory, long-distance dynamics is to be identified through its (regulated) infrared singularities, generically poles in $4 - D$, with D the number of dimensions. When a cross section is factorized in this theory, we can identify the short distance function by a suitable definition and separation of parton distributions, which contain all infrared singularities. From this point of view, our ability of factorize depends on our ability to identify singular long-distance, infrared (IR) behavior in the massless limit. Once identified, we investigate that in each relevant region of phase and loop momentum space, the integrand is either consistent with factorization, or is part of a nonfactorizing class of contributions

that cancels without double counting. In all these arguments, a key tool is that physical infrared singularities of all types are logarithmic in gauge theory cross sections. As a result, a single power suppression in a sum over logarithmically-divergent integrals leads to integrals calculable in perturbation theory.

For renormalizable, non-gauge theories near four dimensions, like ϕ^4 and Yukawa, or ϕ^3 near six dimensions, IR singularities are all "collinear", corresponding to splittings and recombinations of descendants of the two incoming partons, and the task just described is by now well understood. The treatment of collinear singularities in gauge theories, it is worth noting, also requires the treatment of unphysical, longitudinally-polarized vector particles that share the momenta of active partons, at least in covariant gauges. Many of the early all-orders discussions of factorization avoided this complication by working in axial or other physical gauges. A dedicated discussion of the role (and factorization) of unphysical collinear gluons is given, for example, in Ref. [3]. In effective theory treatments, these gluons can be absorbed into the definitions of the physical fields [4, 5]. We shall not pursue this side of factorization here.

For gauge theories, an additional distinguishing feature is soft singularities. In essense, soft gluons can be exchanged between constituents of the incoming hadrons, and such exchanges appear to have the potential to violate causality, and hence factorization.[1] Roughly speaking, it's as if the quarks in proton A could attract the antiquarks in proton B before they collide, even if they approach each other at the speed of light. They do not do so, because until the collision the fields involved are gauge-equivalent to zero field [6], but this feature is obscured in covariant perturbation theory, which incorporates sums over physical and unphysical degrees of freedom. Factorization proofs rely on our ability to organize the phases associated with the external charged particles. The challenge is to disentangle the unphysical component of these phases from the physical gluon-mediated scattering, and to discover when and how the phases cancel in cross sections. At this stage, we emphasize again that all these arguments depend crucially on treating the momenta of incoming particles, the partons and hadrons, as lightlike.

Early all-orders discussions of factorization in hadron-hadron scattering [7, 8, 9, 10, 11, 12] dealt most effectively with collinear singularities, which control evolution, also providing a variety of arguments for the cancellation of soft singularities through virtual-real emission cancellation. This work was inspired in part by a pioneering low-order demonstration of soft-gluon cancellation in Ref. [13] and the older ϕ^4 fragmentation analysis in Ref. [14]. In what follows, I will give an account of the soft gluon interactions in hadronic hard scattering, which set QCD and other gauge theories apart, and were the primary concern in a second round of factorization studies [2] [15, 16, 17, 18].

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[1]Actually, the violation of factorization would be the least of our problems.
[2]For a relatively brief review of the basic arguments, see [19], and for a complete discussion, Ref. [20].

2.1 Soft gluons and phases

We can argue that associated with each parton that arrives at the hard scattering from the past is a nonabelian phase, sensitive only to that component of the gauge field that contracts with the momentum of the parton. Likewise, for the particles that emerge from the hard scattering, their experience of the color fields of other emerging particles is entirely pure gauge, again involving only one component of the gauge field. These phases differ in an important way. This difference illustrates the fundamental relationship between causality and analyticity in quantum field theory, and is a direct consequence of the kinematics of the process. The interactions of fast-moving partons over a semi-infinite time scale, λ, take one of the two general forms, in terms of a "current", labelled here as simply A,

$$
\begin{aligned}
\tilde{A}_+(k \cdot \beta) &= \int_0^\infty d\lambda \, e^{i\lambda k \cdot \beta} A(\lambda\beta) \,, \\
\tilde{A}_-(k \cdot \beta) &= \int_{-\infty}^0 d\lambda \, e^{i\lambda k \cdot \beta} A(\lambda\beta) \,,
\end{aligned}
\tag{3}
$$

where $p^\mu = p^0 \beta^\mu$ defines the light-like velocity, β^μ, $\beta^2 = 0$ of the parton. The function $\tilde{A}_+(k \cdot \beta)$ is analytic in the upper half-plane in $k \cdot \beta_p$, simply because, for $\text{Im}(k \cdot \beta)$ and λ both positive, the integral that defines $\tilde{A}_+$ becomes exponentially suppressed for large, positive λ. Similarly, $\tilde{A}_-(k \cdot \beta)$ is analytic in the lower half-plane of $k \cdot \beta$.

An essential feature of factorization proofs is the wholesale continuation of momentum space contour integrations away from singularities. Along the deformed contours, the "eikonal approximation", $p \cdot k \gg k^2$, holds for generic soft momentum k^μ and high-energy parton momentum p^μ. This condition, in turn, implies that in a subdiagram consisting of collinear lines whose momenta are all of order p, all components of momentum k^μ can be neglected, *except* for $\beta \cdot k$. For lines that terminate at the hard scattering ("active" incoming partons) or originate at the hard scattering (produced high-p_T partons), all potentially "acausal" effects are organized into unitary Wilson line operators (nonabelian phase operators), coming in from the past to the origin, with lightlike velocity β_i, or going out to the future with lightlike velocity β_j,

$$
\begin{aligned}
\Phi_{\beta_i,\text{in}}^{(R_i)}(x) &= \text{P} \, \exp\left[-ig \int_{-\infty}^0 d\lambda \, \beta_i \cdot A^{(R_i)}(\lambda\beta + x) \right] \,, \\
\Phi_{\beta_j,\text{out}}^{(R_j)}(x) &= \text{P} \, \exp\left[-ig \int_0^\infty d\lambda \, \beta_i \cdot A^{(R_i)}(\lambda\beta + x) \right] \,.
\end{aligned}
\tag{4}
$$

Here R is the color representation of the parton in question, and P denotes path ordering. For elastic S-matrix elements in wide-angle scattering, these operators fully describe the interactions of soft gluons, and generate all associated phases. For hard-scattering cross sections, however, the situation is more complicated.

Referring to Fig. 1, integrals over soft gluon attachments to spectator lines can extend from negative to positive infinity on the same line, and hence inolve both Φ_{in} and Φ_{out} operators. This leads to a mismatch in analyticity properties between gluons in the initial
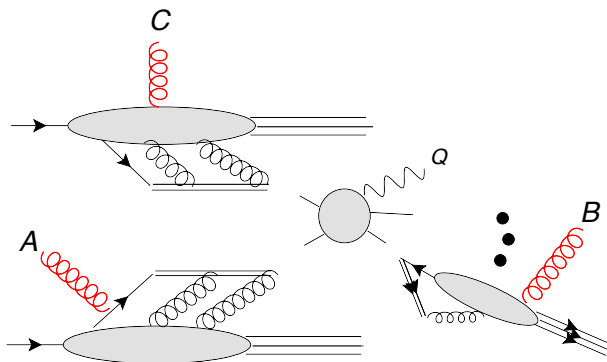
Figure 1: Representation of components of a hard-scattering amplitude in terms of two incoming and a representative outgoing jet in a typical hard-scattering amplitude. The labelled gluons attach to (A) an incoming active parton, (B) a parton emerging from the hard scattering and (C) a spectator parton. Double lines represent the couplings of the unphysically-polarized collinear gluons referred to in the text.

state and those in the final state, and this is a physical effect. This distinction cannot be removed by a gauge transformation, and corresponds physically to gluons that arrive at the time of the hard scattering. Such gluons are the origin of the Glauber gluon problem in hadron-hadron scattering [21, 22]. In essence, Glauber gluons occur in regions of momentum space where momentum contours are pinched between singularities associated with incoming and outgoing phase operators, and for which the eikonal approximation fails. If such a gluon also attaches to the oppositely moving incoming line, we have an essential obstacle to factorization.

It is at this stage that the construction of the cross section comes in. For appropriately-constructed cross sections, final-state phase operators (the Φ_{out}s) cancel in the sum over final states.

The purest example of an outgoing phase is in deep-inelastic scattering, where it is generated by soft gluon interactions with the outgoing quark (and its collinear radiation). This is the origin of the matrix element definition of parton distributions, which can be written for quark field $q_a(x)$ with flavor index a, in the conventional form

$$f_{a/h}(x, \mu) \;=\; \frac{1}{4\pi} \int_{-\infty}^{\infty} d\lambda e^{-i\lambda n \cdot xp} \left\langle h(p) \left| \bar{q}_a(\lambda n)\, \Phi^{\dagger}_{n,\text{out}}(\lambda n, \infty) \frac{n \cdot \gamma}{2} \Phi^{\dagger}_{n,\text{out}}(0, \infty) q_a(0) \right| h(p) \right\rangle .$$

(5)

Here, soft singularities cancel in the final state because of the unitarity of the Wilson line operators, which are both oriented in direction n^{μ}, conventionally chosen as a lightlike direction opposite to the lightlike momentum p of incoming hadron h. Argument μ is the factorization scale, the scale associated with renormalization of the composite operator that defines the matrix element, which becomes ultraviolet divergent because its fields are all separated by lightlike distances ($n^2 = 0$). In effect, the scale μ cuts off the transverse

momenta of radiation. The transverse momentum can actually be fixed in this matrix element to give transverse momentum distributions and factorization, with applications particularly in deep-inelastic scattering and Drell-Yan proceses [23]

We can extend collinear factorization in deeply inelastic scattering far beyond the total cross section, to sums designed to identify jet and related final states, because collinear and soft singularities cancel in partial sums over states. This is a consequence of the "local" unitarity [24, 25, 26] of quantum field theories. This is the origin of the infrared safety of jet cross sections in e^+e^- annihilation, and is closely related to the ideas of effective field theory, and to efforts to reformulate calculations entirely in four dimensions [27]. For example, an interaction Hamiltonian that only excites partons in two directions is not rotationally-invariant, yet it is unitary. As such, summing over its final states gives, by the optical theorem, a forward scattering amplitude that is free of infrared singularities, either soft or collinear [28, 29].

2.2 Unitarity, final states and soft gluons

Moving on to hadron-hadron scattering, the essential observation is that in inclusive cross sections, final state phases cancel in the sum over final states. This will be the case not only for particles emerging from the hard scattering at large p_T, but also for spectator particles that are collinear to the incoming directions.

The "local" unitarity of QCD, mentioned above, enables the cancellation for sectors in momentum space that describe jet cross sections. The long-distance phases that remain are generated only by initial state operators, Φ_{in}. Given what we've seen, this is somewhat disturbing, because it is just the opposite to deep-inelastic scattering, built on Φ_{out} operators. Nevertheless, universal factorization into parton distribution functions, as naturally defined in deep-inelastic scattering, is still possible so long as soft singularities cancel in the hadron-hadron cross section in question. This cancellation was referred to as the transition from "weak" to "strong" factorization in the context of Drell-Yan cross sections, [15, 16, 17, 18]. Reference [30] discussed explicitly how soft radiation sensitive to the difference between the incoming and outgoing Wilson lines cancels.

The technical consequence of this cancellation is that the interactions of all "acausal" initial-state gluons connected to the set of partons collinear to incoming line A and B become coherent, and are summarized by two Wilson lines of the type Φ_{in}. In a fully inclusive cross section, we expect all soft singularities associated with these operators to cancel, enabling a universal factorization. This is the case of Drell-Yan processes. It is natural to ask, however, about less inclusive cross sections. How inclusive is inclusive enough, and what sums over final states will cancel outgoing nonabelian phases up to a momentum scale at which perturbation theory can be applied?

Here is one way of thinking about these questions. We can characterize the length scales associated with the hard scattering by the presence of heavy or high-p_T particles in the final state. This alone, however, does not ensure that final state phases cancel. Rather, a figure of merit corresponding to this cancellation is the coordinate space distance between

the hard scattering in the amplitude and in the complex conjugate. At zero separation, outgoing Wilson lines cancel identically.

The relative displacement of the hard scattering is determined by the range of momenta integrated in the final state, by the usual complementarity of position and momentum measurements. For hard scale Q, if we integrate over a range of momenta in the final state characterized by scales δQ, with δ a fraction such that $\alpha_s(Q) \ln(1/\delta) < 1$, we anticipate that the hard scatterings are localized on the scale $1/(\delta Q)$. Such scales can be used to define jet and related cross sections. In these cases, the final-state phases Φ_{out} corresponding to outgoing partons associated with the Wilson lines will be matched with their hermitian conjugates at scales that are moderately close to $1/(\epsilon Q)$, and we can expect their soft gluons to cancel up to the scale ϵQ, which is consistent with the definition of the cross section. As ϵ decreases, of course, the cross section becomes more and more sensitive to soft radiation. At whatever scale the non cancellation sets in, it is no longer possible to absorb radiation at that energy scale (or higher) into cancelling nonabelian initial state phases, and radiation is no longer incoherent between the descendants of the incoming particles. At scales beyond this one, factorization is broken, in the sense found for amplitudes with forward collinear lines in the final state [31]. Perhaps more accurately, what we have is collinear factorization at an anomalously low factorization scale, determined by the parameters of the set of final states, rather than the hard scattering itself. Beyond this scale, in fact, the fragments of the colliding hadrons, the spectators, influence measured quantities. Important examples that illustrate limitations of collinear and transverse momentum factorization have been identified, for example, in Refs. [32, 33, 34, 35, 36].

Given these general considerations, I'll go on in the next section to give a viewpoint on some of the commonly-discussed cross sections, and the arguments I'm aware of that determine their factorization status.

3   Where we stand

At the risk of introducing evanescent terminology, I'd like to distinguish between "deep" factorization, at the scale of the hard scattering, and "shallow" factorization, in which the factorization scale is set by relatively small (if still perturbative) energy scales that are built into the final state sum in question. It is important to remember, though, that when such scales decrease, corresponding to a less inclusive cross section, with fewer final states, the cross section must decrease. To see this decrease, it will be necessary to generalize the analysis of Sudakov resummation, which can be carried out for fixed-angle exclusive amplitudes [38, 39]. In time, we will learn to love cross sections with shallow factorization, which are in some sense intermediate between inclusive cross sections and exclusive amplitudes. They encode the coherence of color flow in the perturbative regime, in manners that we have not yet fully understood.

3.1 Processes with deep factorization

The arguments above provide, I hope, sufficient framework to interpret at least the claims in the literature of what factorization arguments purport to show. In these terms, a list of processes for which factorization scales are similar to the corresponding scale of the hard scattering includes electroweak annihilation (Drell-Yan, EW boson and Higgs production), and jet production in a wide range of kinematics.

- *Electroweak annihilation (fully inclusive in hadronic final states).* These cross sections, were discussed explicitly in the "second round" of factorization papers, from the mid 1980's [16, 17, 18]. In this case, the cancellation of final states is complete, a claim that is made explicit by the use of time-ordered or light cone-ordered perturbation theory. Later work [30] explored more in detail the manner in which soft radiation associated with the initial state phases cancels, leading to classical "universality" for the parton distributions. Similar results can hold in multiple hard scatterings, where, for example, two electroweak bosons are created at separate partonic interactions in a single collision [40].

- *Inclusive jet production.* Here, we imagine sets of states with jets defined through parameters ϵ with $\alpha_s(Q)ln(1/\epsilon) \sim 1$. For a review, see [41]. The discussion of jet cross sections began early [7], and in Ref. [11] the local unitarity mentioned above was used to show that collinear factorization for jet cross sections of this sort reduces to the Drell-Yan case: soft quanta that connect active or spectator partons to lines in the outgoing jets cancel in the sum over jet final states (assuming that soft radiation is summed inclusively up to a finite fraction of the jet energy scales). This leaves only soft radiation connecting the two incoming jets, just as in Drell-Yan.

- *Single-particle inclusive cross sections*, with inclusivity defined similarly to the jet cross sections defined above provides another example. Arguments for the use of universal fragmentation functions were given in Ref. [42], very much relying on the concepts described above.

- *Multi-particle inclusive cross sections for massive quarks*: As an example, consider final states with a pair of top quarks,. We may observe the top momentum with arbitrary accuracy, so long as we average over the momenta of the anti-top [43]. For cross sections with the hard scale far above the mass of a heavy quark, additional considerations enter [44, 45]. More interestingly, the same cancellations will occur if we measure the transverse momentum of both elements of the top pair, if the pair recoils against sufficiently hard gluonic radiation, over which we average. This is possible, of course, only at an order high enough that such radiation is present in the final state. Thus, we expect it to fail at low orders, and it does [43, 46, 47]. Nevertheless, in the production of very heavy pairs, the cross section is actually dominated by final states with substantial radiation. In this case, the nonradiative portion of the cross section is Sudakov suppressed. In this limit, then, it might be possible to compute the pair transverse momentum self-consistently.

3.2 Shallow factorization

The generic examples of shallow factorization are nonglobal sums over final states. In these cross sections, we impose an energy cutoff in one angular region at a scale much lower than that of unobserved jets and subjets in another region. In general, at high orders these lead to "super-leading" logarithmic enhancements [32, 37], sensitive to both soft and collinear limits, and the factorization scale decreases to the measure of energy flow, rather than the underlying hard scale. Again, in this case, we can think of these logs as due to the non cancellation of outgoing nonabelian phases for energy scales larger than the cutoff imposed to define the cross section.

Going to the limit, if we empty out a section of phase space (say, a complete rapidity gap), we break the matching of nonabelian phases in the amplitude and complex conjugate in hadron-hadron scattering. Nevertheless, in deep inelastic scattering a variant factorization is possible [48], which goes beyond the subject of this contribution.

4 Conclusions

In this review, I've tried to summarize some of the elements of the existing literature on collinear factorization, concentrating on basic intuition and the role of soft radiation. I have left to the side many topics, including transverse momentum factorization [23], and the uses of factorization in resummation. Even purely collinear factorization is a vast subject, and in closing I can only emphasize that even within this narrow focus on a familiar but consequential topic, much important work has surely been neglected. Renewed interest in this subject, partly inspired by the high energy behavior of the spontaeneously broken sectors of the Standard Model [49, 50, 51, 52], will hopefully lead to further insights in how to use examples of "shallow" factorization, to study the transition between short- and long-distance dynamics in gauge theories.

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